2012 Ciphering Time Trials
Solutions

1. **When my secret number is reduced by thirty-seven and this result is then tripled, the result is eighty-seven. What is my secret number?**

   $\frac{87}{3} = 29$ leads to $29 + 37 = 66$.

2. **What is the volume, in cubic meters, of a right rectangular pyramid with a height of 7 m, a width of 6 m, and a length of 5 m?**

   $V_{pyr} = \dfrac{1}{3}Bh = \dfrac{1}{3}wlh = \dfrac{1}{3} \times 7 \times 6 \times 5 = 7 \times 2 \times 5 = 7 \times 10 = 70$

3. **What is the missing term of the sequence beginning 4, 7, 12, 20, 34, 59, 102, ___, 280, …**

   The differences between the terms of this sequence are 3, 5, 8, 14, 25, and 43, the differences of which are 2, 3, 6, 11, 18, the differences of which are 1, 3, 5, 7. The next two differences at this level appear to be 9 and 11, making the next level up 27 and 38, making the next level up 70 and 108, making the sequence itself 172 and 280, for an answer of 172.

4. **What number is twenty-three less than the product of nineteen and thirty-two?**

   $19 \times 32 = 20 \times 32 - 32 = 640 - 32 = 608$ and $608 - 23 = 585$.

5. **If $c(d) = 3 - 2^{4d-5}$, evaluate $c(3)$.**

   $c(3) = 3 - 2^{4\times3-5} = 3 - 2^{12-5} = 3 - 2^7 = 3 - 128 = -125$

6. **The data set $\{4, 9, 10, 17, 19\}$ also contains positive integers *x* & *y*. If the mean is greater than the median which is greater than the unique mode, what is the smallest possible sum of *x* and *y*?**

   If both x and y are small, the median will be 9, which would allow the mode to be 4 (or lower if $x = y$). For the mean to be greater than 9, the sum $4 + 9 + 10 + 17 + 19 + x + y = 59 + x + y$ must be greater than $7 \times 9 = 63$, so that $x + y > 4$, making 5 the answer. Specifically, 1 and 4 can be *x* and *y*.

7. **What is the equation, in slope-intercept $(y = mx + b)$ form, of the line through the points $(3, -7)$ and $(-2, -4)$?**

   The slope is $\frac{-4-(-7)}{-2-3} = \frac{-4+7}{-5} = -\frac{3}{5}$, so the equation is $y = -\frac{3}{5}x + b$. Substituting the second point gives $-4 = -\frac{3}{5}(-2) + b$, then $-\frac{20}{5} = \frac{6}{5} + b$, so that $b = -\frac{26}{5}$ for an answer of $y = -\frac{3}{5}x - \frac{26}{5}$.

8. **What is the area, in square meters, of a right triangle with a leg measuring 10 m and a hypotenuse measuring 14 m?**

   This triangle has dimensions twice those of a triangle with a leg of 5 and a hypotenuse of 7, which will have another leg measuring $\sqrt{7^2 - 5^2} = \sqrt{49 - 25} = \sqrt{24} = 2\sqrt{6}$, for an answer of $\frac{1}{2} \times 10 \times 4\sqrt{6} = 20\sqrt{6}$.

9. What are the coordinates, in the form (x, y), of the center of the locus of points satisfying $4x^2 + 3y^2 - 9x + 12y = 100$?

Completing the squares gives $4\left(x^2 - \frac{9}{4}x\right) + 3(y^2 + 4y) = 100$, then $4\left(x - \frac{9}{8}\right)^2 + 3(y+2)^2 = 100 + \cdots$, so that the center is at the point $\left(\frac{9}{8}, -2\right)$.

10. Simplify by multiplying and combining like terms: $(4x^3 + 12x + 2)(x^2 - 3)$

$$\begin{aligned}(4x^3 + 12x + 2)(x^2 - 3) &= 4x^5 + 12x^3 + 2x^2 - 12x^3 - 36x - 6 \\ &= 4x^5 + 2x^2 - 36x - 6\end{aligned}$$

11. What is the sum of the positive two-digit odd integers?

The sum of the *n* smallest odd integers is n^2, so our answer is $50^2 - 5^2 = 2500 - 25 = 2475$.

12. When three cards are drawn from a standard 52-card deck, what is the probability that there are at least two cards of the same rank (e.g. two Kings and a Three)?

This probability is equal to one minus the probability that the three cards are different ranks. The first card has a probability of 1, the second card has a probability of $\frac{48}{51} = \frac{16}{17}$, and the third card has a probability of $\frac{44}{50} = \frac{22}{25}$, so that our answer becomes $1 - \frac{16}{17} \times \frac{22}{25} = 1 - \frac{352}{425} = \frac{73}{425}$.

13. Evaluate: $90817 - 26354$

$90817 - 26354 = 64463$

14. What is the measure, in degrees, of an angle that is supplementary to an angle that is its own complement?

Complementary angles sum to 90°, so a 45° angle is its own complement. Supplementary angles sum to 180°, for an answer of $180 - 45 = 135$.

15. Liana's marble collection contains one marble in each of many different sizes. The smallest marble has a diameter of one millimeter, while the largest marble has a diameter of 1.5 centimeters. If each marble has a diameter one millimeter larger than the next smaller marble, what is the ratio of the weight of the entire marble collection to the weight of the smallest marble?

The smallest marble has a weight proportional to its volume, which is proportional to the cube of its radius, which is $1^3 = 1$. The total weight (and volume) of the marbles will be proportional to the sum of the cubes from 1^3 to 15^3. The sum of the first *n* cubes is the square of the first n counting numbers, which is $\frac{n(n+1)}{2}$, for an answer of $\left(\frac{15 \times 16}{2}\right)^2 = (8 \times 15)^2 = 120^2 = 14400$.

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16. What are the coordinates, in the form (x, y) of the x-intercept(s) of the parabola $y = 2x^2 - 3x - 20$?

The equation factors to $y = (2x+5)(x-4)$ with zeros at $x = -\frac{5}{2}$ and $x = 4$, for intercepts at $\left(-\frac{5}{2}, 0\right)$ and $(4,0)$.

17. Two points are chosen inside a square with a perimeter of 160 meters. What is the probability that both points are within ten meters of a side of the square? Note: the two points do not need to be close to the *same* side of the square.

Neither point can be in a central square with 20 m sides, for a probability of $\left(1 - \frac{20}{40} \times \frac{20}{40}\right)^2 = \left(1 - \frac{1}{2} \times \frac{1}{2}\right)^2 = \left(1 - \frac{1}{4}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

18. If $m(n) = \frac{(m+2)(3-4m)}{5m-6}$, evaluate $m'(1)$.

Using the quotient, product, and chain rules a little carefully gives
$m'(n) = \frac{(5m-6)[(m+2)(-4)+(1)(3-4m)]-(m+2)(3-4m)(5)}{(5m-6)^2}$, so that
$m'(1) = \frac{(-1)[(3)(-4)+(1)(-1)]-(3)(-1)(5)}{(-1)^2} = (-1)[-12-1] + 15 = 13 + 15 = 28.$

19. Express in simplest radical form: $\sqrt{832}$

$\sqrt{832} = \sqrt{8 \times 104} = 2\sqrt{2 \times 4 \times 26} = 2 \times 2\sqrt{2 \times 2 \times 13} = 2 \times 4\sqrt{13} = 8\sqrt{13}$

20. What is the circumference, in meters, of a circle inscribed in an isosceles triangle with sides measuring 5 m, 5 m, and 8 m?

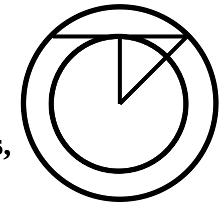
The isosceles triangle can be divided into two congruent right triangles, each with a hypotenuse of 5 and a leg of 4, making the other leg (the height of the original triangle) 3. This means that the isosceles triangle's area is $3 \times 4 = 12 = \frac{1}{2} \times r \times (5+5+8) = 9r$, so that $r = \frac{12}{9} = \frac{4}{3}$, for a circumference of $C = 2\pi r = 2\pi \times \frac{4}{3} = \frac{8\pi}{3}$.

21. Evaluate the determinant: $\begin{vmatrix} -2 & 3 & 0 \\ 0 & 1 & -3 \\ -1 & 2 & 1 \end{vmatrix}$

$$\begin{vmatrix} -2 & 3 & 0 \\ 0 & 1 & -3 \\ -1 & 2 & 1 \end{vmatrix} = (-2)[1 \times 1 - 2(-3)] + (-1)[3(-3) - 1 \times 0] = (-2)7 - (-9)$$
$$= -14 + 9 = -5$$

22. How many liters of a 30% acid solution should be combined with six liters of a 50% acid solution to create a 35% acid solution?

Because $\frac{35-30}{50-35} = \frac{5}{15} = \frac{1}{3}$, there is three times as much 30% solution as 50% solution, for an answer of 18.



23. Two circles have the same center, and a 12-meter chord of the larger circle is tangent to the smaller circle. What is the area, in square meters, of the area between the two circles?

In the figure to the right, you can see that $R^2 - r^2 = 6^2 = 36$, while the area we're seeking is $\pi R^2 - \pi r^2 = \pi \times 36 = 36\pi$.

24. In the cryptarithm below where each instance of a letter represents the same digit (0-9) and different letters represent different digits, what is the largest possible value of the five-digit number *ABCDE*?

$$\begin{array}{r} \boldsymbol{ABCD} \\ \underline{\boldsymbol{-DEC}} \\ \boldsymbol{CDC} \end{array}$$

Clearly, $A = 1$. After this, we'd like to try $B = 9$ to get a large answer, but there's no value of D that would cause borrowing in this case. Trying $B = 8$ would require $D = 9$, but then C would have to be 9 or 8, both of which are already in used. Trying $B = 7$ would require $D = 9$ or $D = 8$, with $C = 8$ in the first case or $C = 9$ in the second. The one's digit's $D - C = C$ does not work in the first case, but does in the second, causing $E = 0$ and an answer of 17980.

25. Evaluate: $\boldsymbol{4^{\left(3^{\left(2^{\left(1^0\right)}\right)}\right)}}$

$$4^{\left(3^{\left(2^{\left(1^0\right)}\right)}\right)} = 4^{\left(3^{\left(2^1\right)}\right)} = 4^{\left(3^2\right)} = 4^9 = 2^{18} = 2^{10} \times 2^8 = 1024 \times 256 = 262144$$

26. Evaluate as a base seven number: $\boldsymbol{3456_7 - 2514_7}$

Don't convert, just do the arithmetic as usual, except that when you borrow, you'll borrow 7, not 10. $6 - 4 = 2$, $5 - 1 = 4$, $4 - 5$ becomes $11 - 5 = 6$ after borrowing, and $3 - 2$ becomes $2 - 2 = 0$ after borrowing, for an answer of 642_7.

27. When the last group of three Trick-or-Treaters came by, Kelly had six Mee Wusketeers and eight Thrilky May candy bars left, and he wanted to give all of them away so that he wouldn't end up eating the leftovers. If he doesn't care about fairness, in how many distinguishable ways can Kelly pass out the candy bars?

Consider putting the each kind of candy bar in a different bowl, along with two "switch kids" bars. As you draw bars one at a time, you'll keep giving them to the same kid until you get a "switch kids" bar, at which point you'll start giving bars to the next kid. Doing this, there are $\binom{6+2}{2} = \binom{8}{2} = \frac{8\times7}{2} = 4 \times 7 = 28$ ways to pass out the Mee Wusketeers bars and $10c2 = \frac{10\times9}{2} = 5 \times = 45$ ways to pass out the Thrilky May bars, for a total of $28 \times 45 = 14 \times 90 = 1260$ ways to pass out both kinds of candy bars.

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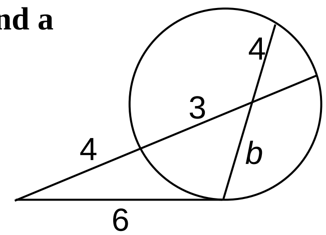
28. My piggy bank contains 37 coins, each of which is either a nickel, a dime, or a quarter. If the total value of the coins is \$4.70 and the number of nickels is one more than the sum of the numbers of dimes and quarters, how many quarters are there?

We can write $n + d + q = 37$ and $d + q = n - 1$, which combine to give $2n - 1 = 37$, with a solution of $n = 19$. Thus, there are 19 nickels and 18 of some combination of dimes and quarters. The nickels are worth $5 \times 19 = 95$ cents, so the dimes and quarters are worth a total of $4.70 - .95 = 3.75$. If the 18 were all dimes, they would be worth \$1.80, which is $\$3.75 - \$1.80 = \$1.95$ less than what we want. Each time we turn a dime into a quarter, we gain $25 - 10 = 15$ cents, so we must do this $\frac{1.95}{.15} = \frac{195}{15} = 13$ times, and there are thus 13 quarters.

29. In the figure to the right containing a circle, a chord, a secant, and a tangent, distances are given in meters. What is the value of *b*?



The secant and tangent allow us to write $6^2 = 4(4 + 3 + a)$, which becomes $9 = 7 + a$ and yields $a = 2$, where a is the missing length of the secant. Now the chords allow us to write $3 \times 2 = 4x$, which yields $x = \frac{6}{4} = \frac{3}{2}$.

30. A right triangle has a hypotenuse measuring 12 m and a leg measuring 9 m. What is the cosecant of the smallest angle in the triangle?

This triangle is three times a triangle with a hypotenuse of 4 and a leg of 3, which will have another leg of $\sqrt{4^2 - 3^2} = \sqrt{16 - 9} = \sqrt{7}$. This triangle will have the same angles and thus the same cosecant, so we'll just work with it. The smallest angle will be across from the smallest side, which is $\sqrt{7}$, and the cosecant is the reciprocal of the sine, so the hypotenuse over the opposite side, giving $\frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$.