2012 Four-by-Four Competition
Solutions

1. **What is the sum of the terms of an infinite geometric sequence with first term 2016 and common ratio $\frac{1}{4}$?**

   $$S = \frac{a}{1-r} = \frac{2016}{1-\frac{1}{4}} = \frac{2016}{\frac{3}{4}} = 2016 \times \frac{4}{3} = 672 \times 4 = 2688$$

2. **In how many distinguishable ways can five extra copies of your "sided" housekey be arranged on a keyring if they are colored red, orange, yellow, green, and green? You can tell if two "sided" keys are oriented the same way or opposite ways on the keyring.**

   There are $(5-1)! = 4! = 24$ ways to arrange five things in a circle, but because two of our things are indistinguishable, we must divide by $2! = 2$ to get $24 \div 2 = 12$ ways to arrange the keys around the keyring. Then, if the red key is pointed one way, there are two choices for the orientation of each other key, for an answer of $12 \times 2^4 = 12 \times 16 = 192$.

3. **If you can buy K kilograms of corn meal with Q quarters, how many dimes would it take to buy 5 kilograms of corn meal? Express your answer in a reduced fractional form.**

   We originally spent Q quarters, which would be $Q \times \frac{25}{10} = \frac{5Q}{2}$ dimes. We're multiplying the number of kilograms by $\frac{5}{K}$, so we need to multiply the dimes similarly, getting $\frac{5Q}{2} \times \frac{5}{K} = \frac{25Q}{2K}$. This might not be an integer, so you really ought to round it up if necessary: $\left\lceil \frac{25Q}{2K} \right\rceil$.

4. **A pentagon has integer side lengths k, m, n, p, q (not necessarily arranged in that order) when measured in meters. The ratio of m to n is 5, the ratio of p to q is 7, and no two sides are congruent. If $m, n, p, \& q$ are all less than 100, what is the largest possible value of k?**

   To get the largest fifth side, we'd like four other large sides. $m \& n$ could be 19 & 95, while $p \& q$ could be 14 & 98. Similar to the Triangle inequality, the largest possible value of k would be $19 + 95 + 14 + 98 - 1 = 225$.

5. **If $r(s) = s^3 - s - 1$ and $t(u) = u + u^2$, evaluate $r\left(t\left(t(r(1))\right)\right)$.**

   $r(1) = 1^3 - 1 - 1 = 1 - 2 = -1$, $t(-1) = -1 + (-1)^2 = -1 + 1 = 0$, $t(0) = 0 + 0^2 = 0 + 0 = 0$, and $r(0) = 0^3 - 0 - 1 = 0 - 1 = -1$.

6. **Evaluate: $-9 - (-8)(-7)(-6)^{-2} - 5$**

   $$-9 - 5 - \frac{8 \times 7}{6 \times 6} = -14 - \frac{2 \times 7}{3 \times 3} = -14 - \frac{14}{9} = \frac{-9 \times 14 - 14}{9} = -\frac{10 \times 14}{9} = -\frac{140}{9}$$

7. **What is the area of the convex quadrilateral with vertices at the points $(2, 3)$, $(3, -4)$, $(4, 5)$, and $(10, -6)$?**

   When you think about the points, you realize that they are in the order $(2,3)$, $(4,5)$, $(10, -6)$, $(3, -4)$, which allows us to use the Shoelace Method to create a left-hand sum of $3 \times 4 + 5 \times 10 + (-6) \times 3 + (-4) \times 2 = 12 + 50 - 18 - 8 = 62 - 26 = 36$ and a right-hand sum of $2 \times 5 + 4 \times (-6) + 10 \times (-4) + 3 \times 3 = 10 - 24 - 40 + 9 = 19 - 64 = -45$.
   Subtracting and dividing by two gives $\frac{36-(-45)}{2} = \frac{36+45}{2} = \frac{81}{2}$.

8. **What is the smallest positive integer greater than 1000 with exactly four distinct prime factors?**

   If the prime factors are 2, 3, 5, & 7, the number must be a multiple of $2 \times 3 \times 5 \times 7 = 210$, the smallest of which is $5 \times 210 = 1050$. If the prime factors are 2, 3, 5, & 11, the number must be a multiple of 330, which doesn't work as well. 2, 3, 5, 13 gives multiples of 390, also not good. 2, 3, 5, 17 gives 510, leading to 1020. There's nothing better than this, but you could spend more time confirming that.

9. **What is the perimeter, in meters, of a heptagon with sides measuring 13 m each?**

   $7 \times 13 = 70 + 21 = 91$

10. **In a five-element data set of integer test scores from 0 to 100 inclusive, the mean is 73. What is the smallest possible value of the median?**

    If the mean is 73, the sum is $5 \times 73 = 350 + 15 = 365$. For a small median, we want three small elements and two large elements, so we can choose x, x, x, 100, 100 as the set, for a sum of $200 + 3x = 365$. This becomes $3x = 165$, for an answer of $x = 55$.

11. **In the system $d + f + g + 4h = 5$, $d + f + 4g + h = 6$, $d + 4f + g + h = 7$, and $4d + f + g + h = 8$, what is the value of d?**

    Adding the four equations gives $7d + 7f + 7g + 7h = 30$, so that $d + f + g + h = \frac{26}{7}$.
    Subtracting this from the final equation gives $3d = 8 - \frac{30}{7} = \frac{56-26}{7} = \frac{30}{7}$, so that $d = \frac{30}{7} \div 3 = \frac{10}{7}$.

12. **Evaluate: $203^3 - 197^3$**

    $$203^3 - 197^3 = (203 - 197)(203^2 + 203 \times 197 + 197^2)$$
    $$= 6\big((203 + 197^2) - 203 \times 197\big) = 6\big(400^2 - (200 + 3)(200 - 3)\big)$$
    $$= 6(160000 - 200^2 + 3^2) = 6(160000 - 40000 + 9) = 6 \times 120009$$
    $$= 720054$$

13. **Evaluate: $\left(4096^{2^{-3}}\right)^4$**

    $$\left(4096^{1/8}\right)^4 = 4096^{1/2} = (2^{12})^{1/2} = 2^6 = 64$$

14. What is the distance between the reflection of the origin across the line $y = 7$ and the reflection of the point $(2, 4)$ across the line $x = -5$?

The origin, (0,0), is seven units from $y = 7$, so the reflection will be, too, so it will be at (0,14). (2,4) is seven units from $x = -5$, so its reflection will be $(-12,4)$. The distance between these points will be

$$\sqrt{(0 - (-12))^2 + (14 - 4)^2} = \sqrt{12^2 + 10^2} = 2\sqrt{6^2 + 5^2} = 2\sqrt{36 + 25} = 2\sqrt{61}.$$

15. Set G is the set of positive three-digit integers that are factors of one billion, and Set H is the set of positive integer multiples of one-hundred less than ten thousand. How many elements are in the set $H \cap G'$?

There are 99 elements in set H, but we need to exclude those that are in G, which is are 100, 200, 400, 500, and 800 (5 elements), for an answer of $99 - 5 = 94$.

16. Evaluate: $\int_0^1 m e^{2m+1} dm$

Integrating by parts, let $u = m$ and $dv = e^{2m+1} dm$, so that $du = dm$ and $v = \frac{1}{2} e^{2m+1}$.

Now we want $uv - \int v du = \frac{1}{2} m e^{2m+1} - \int \frac{1}{2} e^{2m+1} dm = \frac{1}{2} m e^{2m+1} - \frac{1}{4} e^{2m+1}$. Evaluating at 1 and 0 and subtracting gives $\frac{1}{2} e^3 - \frac{1}{4} e^3 - \left(0 - \frac{1}{4} e\right) = \frac{1}{4} e^3 + \frac{1}{4} e$.

17. What is the tens digit of 12^{12}?

$12^{12} = (10 + 2)^{12} = 2^{12} + 12 \times 10^1 \times 2^{11} + \frac{12 \times 11}{2} \times 10^2 \times 2^{10} + \cdots$, with all subsequent terms having tens & ones digits of 0. This becomes $4096 + 12 \times 20480$, and only the last two digits matter, for an answer based on $96 + 60 = 156$, for a tens digit of 5.

18. What is the surface area, in square meters, of a right circular cylinder with a base radius of 8 m and a height of 5 m?

The surface area of a cylinder is
$SA = 2(\pi r^2) + (2\pi r)h = 2\pi \times 8^2 + 2\pi \times 8 \times 5 = 128\pi + 80\pi = 208\pi.$

19. Evaluate: $\frac{2! \times 4! \times 6! \times 8!}{10!}$

$$\frac{2! \times 4! \times 6! \times 8!}{10!} = \frac{2! \times 4! \times 6!}{10 \times 9} = \frac{2 \times 4 \times 3 \times 2 \times 6 \times 5 \times 4 \times 3 \times 2}{10 \times 9} = 4 \times 2 \times 6 \times 4 \times 2$$
$$= 64 \times 6 = 384$$

20. If $\sin j = \frac{3}{5}$ and $\cos k = -\frac{5}{13}$, where j and k lie in the same quadrant, evaluate $\tan(j - k)$.

If the sine is positive and the cosine is negative, the angles must be in the second quadrant.

$$\tan(j - k) = \frac{\tan j - \tan k}{1 + \tan j \tan k} = \frac{\left(-\frac{3}{4}\right) - \left(-\frac{12}{5}\right)}{1 + \left(-\frac{3}{4}\right)\left(-\frac{12}{5}\right)} = \frac{\frac{-15+48}{20}}{\frac{20+36}{20}} = \frac{33}{56}$$

21. What value(s) of n satisfy $9n - 87 = 6n + 54$?

$3n = 54 + 87 = 141$ becomes $n = 47$.

22. An equilateral triangle is inscribed in a circle, and this circle is then inscribed in a larger equilateral triangle. What is the ratio of the area of the smaller triangle to that of the larger triangle, expressed as a fraction?

Orienting the triangles so that one points up and one points down, it should be clear that the smaller triangle is $\frac{1}{4}$ the area of the larger triangle.

23. I have 6567 identical gold coins, but I know that one of them is counterfeit, and very slightly heavier than the others. Fortunately, I have a very accurate balance that I can use to compare the weight of up to 1000 coins with any other group of up to 1000 coins. If I use it wisely, what is the greatest number of times I must be prepared to use the balance to find the counterfeit coin?

Weighing 1000 against 1000, the worst thing you could learn is that the heavy coin is in the 4567 you didn't weigh. Another 1000 vs. 1000 could narrow it to 2567. 856 vs. 856 could put it in an 856, then 285 vs. 285 could put it in 286. 95 vs. 95 could put it in 96, then 32 vs 32 will put it in 32. 11 vs. 11 could put it in 11, then 4 vs. 4 could put it in 4. 1 vs. 1 could put it in , and then 1 vs. 1 will tell you which coin it is, for a total of 10 weighings.

24. In what base w will 12345_w equal 3267_{10}?

12345_w is larger than w^4. $5^4 = 25^2 = 625$, $6^4 = 36^2 = 1296$, $7^4 = 49^2 < 2500$, and $8^4 = 64^2 = 4096$. w cannot be 8, but might be 7, in which case $12345_w = 5 + 4 \times 7 + 3 \times 49 + 2 \times 343 + 2401 = 5 + 28 + 147 + 686 + 2401 = 180 + 3087 = 3267$.

25. When a fair coin is flipped repeatedly, what is the probability that the fifth head occurs on the eighth flip?

The probability that the eighth flip is heads is $\frac{1}{2}$. The probability that exactly four of the first seven flips were heads is $7c4 \times \left(\frac{1}{2}\right)^7 = \frac{7 \times 6 \times 5}{3 \times 2} \times \frac{1}{2^7} = \frac{7 \times 5}{1} \times \frac{1}{2^7} = \frac{35}{128}$, for an answer of $\frac{35}{128} \times \frac{1}{2} = \frac{35}{256}$.

26. A regular polygon has interior angles that are five times the size of its exterior angles. How many diagonals does this polygon have?

The exterior angle must be $\frac{180}{6} = 30 = \frac{306}{12}$, so the polygon has 12 sides and $\frac{12(12-3)}{2} = \frac{12 \times 9}{2} = 6 \times 9 = 54$ diagonals.

27. Evaluate in terms of $i = \sqrt{-1}$: $(1 + 2i + 3i^2 + 4i^3)(6i^2 + 7i^3 - 8i^4)$

$(1 + 2i - 3 - 4i)(-6 - 7i - 8) = (2 + 2i)(14 + 7i) = 28 + 14i + 28i - 14 = 14 + 42i$

28. If $a + 2b + 3c = 2a - 3b + c = 3a + 2b - c - 1$, what is the value of $a + b - 2c$?

There are three equations in this relationship. The first is $a + 2b + 3c = 2a - 3b + c$, giving $a - 5b - 2c = 0$. The second is $2a - 3b + c = 3a + 2b - c - 1$, giving $a + 5b - 2c = 1$. The third is $a + 2b + 3c = 3a + 2b - c - 1$, giving $2a - 4c = 1$. Subtracting the first two results gives $10b = 1$, so $b = \frac{1}{10}$. Adding the first two results gives $2a - 4c = 1$, which is the same as the third results. This would be a problem, except that if we divide it by two we get $a - 2c = \frac{1}{2}$, which is most of our answer. $\frac{1}{2} + \frac{1}{10} = \frac{5}{10} + \frac{1}{10} = \frac{6}{10} = \frac{3}{5}$.

29. On the first three days of their road trip, Cherie's family drove 500 miles in 8 hours, 400 miles in 9 hours, and 300 miles in 3 hours. What was their average driving speed, in miles per hour, over those three days?

The average speed is the total distance over the total time: $\frac{500+400+300}{8+9+3} = \frac{1200}{20} = 60$.

30. In the recursive sequence defined by $d_0 = 16$ and $d_{f+1} = \left(\frac{d_f}{2}\right)^2 - d_f$, what is the value of d_3?

$d_1 = \left(\frac{16}{2}\right)^2 - 16 = 8^2 - 16 = 64 - 16 = 48$,

$d_2 = \left(\frac{48}{2}\right)^2 - 48 = 24^2 - 48 = 576 - 48 = 528$, and

$d_3 = \left(\frac{528}{2}\right)^2 - 528 = 264^2 - 528 = 69696 - 528 = 69168$

31. The vertices of a regular polygon are lettered consecutively in clockwise order starting with A, B, C, … If the line through M and F passes through the center, the line through A and what other vertex will also pass through the center?

From G, H, I, J, K, L, M is seven steps. Seven steps from A is B, C, D, E, F, G, H.

32. What is the sum of the positive four-digit integers that leave a remainder of 3 when divided by 7?

$1000 \div 7 = 142r6$ and $10000 \div 7 = 1428r4$, so the desired numbers range from $997 + 7 = 1004$ to 9999. This means there are $\frac{\left(\frac{9999-1004}{7}+1\right)}{2} = \frac{\left(\frac{8995}{7}+1\right)}{2} = \frac{(1285+1)}{2} = \frac{1286}{2} = 643$ pairs that each sum to $9999 + 1004 = 11003$, for a total of $11003 \times 643 = 6430000 + 643000 + 1929 = 7,074,929$.

33. Express in simplest radical form: $\sqrt[5]{448}$

$\sqrt[5]{448} = \sqrt[5]{2^2 \times 112} = \sqrt[5]{2^4 \times 28} = \sqrt[5]{2^6 \times 7} = 2\sqrt[5]{2 \times 7} = 2\sqrt[5]{14}$

34. What value(s) of j satisfy $\frac{j-1}{j^2-2j-3} = \frac{j+4}{j^2+3j-2}$?

Cross-multiplying gives $j^3 + 2j^2 - 5j + 2 = j^3 + 2j^2 - 11j - 12$, which becomes $-5j + 2 = -11j - 12$, then $6j = -14$, finally giving $j = -\frac{14}{6} = -\frac{7}{3}$.

35. In how many ways can I distribute 20 food pellets into five distinguishable fish tanks so that every tank gets at least one pellet and no two tanks receive the same number of pellets?

We can give 1, 2, 3, 4, & 10 or 1, 2, 3, 5, & 9 or 1, 2, 3, 6, & 8 or 1, 2, 4, 5, & 8 or 1, 2, 4, 6, & 7 or 1, 3, 4, 5, & 7 or 2, 3, 4, 5, & 6. Each of these 7 can be distributed in $5! = 120$ ways, for a total of $7 \times 120 = 840$.

36. I have a circular rug that just barely fits in a square room of my house. I'd like to order four custom-made circular rugs to fit in the corners of this room so that they'll barely touch the two walls and the central rug. If my room measures 8 m on a side, what diameter, in meters, should I instruct the rug-maker to make the four rugs?

If you draw the square room, the big rug, a little rug, and a diagonal through them all, you can write $4\sqrt{2} - 4 = r + r\sqrt{2}$, so that $r = \frac{4\sqrt{2}-4}{1+\sqrt{2}} = \frac{(4\sqrt{2}-4)(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})} = \frac{4\sqrt{2}+4\sqrt{2}-4-8}{-1} = 12 - 8\sqrt{2}$ and the diameter is $2(12 - 8\sqrt{2}) = 24 - 16\sqrt{2}$.

37. What is the name of the Platonic Solid on which four equilateral triangles meet at each vertex?

You should know that this shape is an "octahedron", the solid formed when the centers of the faces of a cube are used as vertices of a new solid.

38. Evaluate: $12345.6789 + 2.468 + 86420 + 9.7531$

Lining up the decimal points, $12345 + 2 + 86420 + 9 = 98776$ and $.6789 + .468 + .7531 = 1.9$ for a total of $98776 + 1.9 = 98777.9$.

39. What is the sum of the squares of the roots of $v^3 - 2v^2 + 3v - 4 = 0$?

We want $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ac) = 2^2 - 2 \times 3 = 4 - 6 = -2$.

40. A bag of marbles contains three red and seven blue marbles. My friend and I take turns drawing one marble from the bag, with my friend going first, and neither of us replacing our marbles. What is the probability that I draw the first red marble?

I could in on any of my first four turns:

$\frac{7}{10} \times \frac{3}{9} + \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{3}{7} + \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} + \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} = \frac{7}{10} \times \frac{1}{3} \times \frac{4}{4} + \frac{1}{10} \times \frac{3}{3} \times \frac{5}{4} + \frac{1}{10} \times \frac{1}{3} \times \frac{1}{4} \times \frac{2}{1} \times \frac{3}{1} + \frac{1}{10} \times \frac{1}{3} \times \frac{1}{4} = \frac{28+15+6+1}{120} = \frac{50}{120} = \frac{5}{12}$.