2013 Team Scramble
Solutions

1. Evaluate: $\mathbf{9754321 - 2356789}$

The standard subtraction algorithm gives $9754321 - 2356789 = 7{,}397{,}532$.

2. Evaluate: $\mathbf{987654321 \div 289}$

The standard division algorithm gives $987654321 \div 289 = 3{,}417{,}489$.

3. What number is 37 more than twice the number that is 41 less than the product of 53 and 59?

$2(53 \times 59 - 41) + 37 = 2(3127 - 41) - 37 = 2 \times 3086 - 37 = 6172 - 37 = 6{,}209$

4. Evaluate as a mixed number: $\mathbf{7\frac{5}{6} \times 4\frac{8}{9}}$

$$\frac{47}{6} \times \frac{44}{9} = \frac{47}{3} \times \frac{22}{9} = \frac{1034}{27} = 38\frac{8}{27}$$

5. Evaluate: $\mathbf{-1(-2 - 3^{-2})^{-1}(-2) - (-3) - 2^{-1}}$

$$= 2\left(-2 - \frac{1}{9}\right)^{-1} + 3 - \frac{1}{2} = 2\left(-\frac{19}{9}\right)^{-1} + \frac{5}{2} = -\frac{18}{19} + \frac{5}{2} = \frac{-36 + 95}{38} = \frac{59}{38}$$

6. How many hours are there in a leap year?

$24 \times 366 = 8784$

7. Evaluate: $\mathbf{\log_3 4 \times \log_5 27 \times \log_{256} 25}$

$$= \frac{\ln 4}{\ln 3} \times \frac{\ln 27}{\ln 5} \times \frac{\ln 25}{\ln 256} = \frac{2\ln 2}{\ln 3} \times \frac{3\ln 3}{\ln 5} \times \frac{2\ln 5}{8\ln 2} = \frac{2 \times 3 \times 2}{8} = \frac{3}{2}$$

8. Evaluate: $\mathbf{6 + (8 \times 9 - 4^2) \div 2}$

$$= 6 + (72 - 16) \div 2 = 6 + \frac{56}{2} = 6 + 28 = 34$$

9. Evaluate: $\mathbf{1234^2 - 123^2}$

$= (1234 - 123)(1234 + 123) = 1111 \times 1357 = 1{,}507{,}627$

10. Arrange the letters in order of ascending value:
A = the number of days in March **B = the number of inches in a yard**
C = the number of diagonals in a decagon **D = the number of weeks in a year**

$A = 31$, $B = 3 \times 12 = 36$, $C = \frac{10\times 7}{2} = 5 \times 7 = 35$, and $D = 52$, so the answer is ACBD.

11. What value(s) of *f* satisfy $\mathbf{29f + 38 = 2329}$?

$29f = 2329 - 38 = 2291$, so $f = 79$.

12. Simplify by multiplying and combining like terms: $(2h+3)(4h-5)(6-7h)$

$= (8h^2 + 2h - 15)(6 - 7h) = -56h^3 + 34h^2 + 117h - 90$

13. What ordered pair, in the form (j,k), satisfies the equations $4k - 3j = 56$ and $2j + 5k = 1$?

Adding twice the first equation to three times the second gives $23k = 115$, so $k = 5$, and thus $j = -12$, for an answer of $(-12,5)$.

14. If Matt can eat a pizza in an hour and Tom can eat a pizza in 45 minutes, how many minutes would it take the two of them to eat a pizza together?

Their speeds are $\frac{1}{60}$ and $\frac{1}{45}$ of a pizza per minute, and can be added to give a combined speed of $\frac{1}{60} + \frac{1}{45} = \frac{3}{180} + \frac{4}{180} = \frac{7}{180}$. Thus, it would take them $\frac{1}{\frac{7}{180}} = \frac{180}{7}$ minutes.

15. On average, three people can remodel seven bathrooms in 24 days. On average, how many people would be needed to remodel 28 bathrooms in eight days?

The original job took 3 people, but the new job requires doing $\frac{28}{7} = 4$ times as much work in $\frac{8}{24} = \frac{1}{3}$ of the time, so we'll need $\frac{3\times 4}{\frac{1}{3}} = 12 \times 3 = 36$ people.

16. Aaron bikes 50 km at 30 km/hr, then takes another hour to ride 20 km. What was his average speed in km/hr over his entire ride?

His average speed will be his total distance divided by the total time it took, which is $\frac{50+20}{\frac{50}{30}+1} = \frac{70}{\frac{8}{3}} = \frac{210}{8} = \frac{105}{4}$.

17. In which quadrant does the point $(7,-1)$ lie?

This is just something you memorize; the lower right quadrant is the fourth quadrant.

18. What is the equation, in slope-intercept form, of the line through the point $(2,3)$ and perpendicular to the line $-3x + 5y = 987$?

The perpendicular line will be of the form $5x + 3y = C$. Substituting the desired point gives $D = 5x + 3y = 5 \times 2 + 3 \times 3 = 10 + 9 = 19$, so the line is $5x + 3y = 19$. In slope-intercept form, this line is $y = -\frac{5}{3}x + \frac{19}{3}$.

19. What is the distance between the point $(-4,-9)$ and the midpoint of the line segment from the point $(4,-7)$ to the point $(-2,19)$?

The midpoint is $\left(\frac{4+(-2)}{2}, \frac{-7+19}{2}\right) = \left(\frac{2}{2}, \frac{12}{2}\right) = (1,6)$, so the distance is $\sqrt{(-4-1)^2 + (-9-6)^2} = \sqrt{5^2 + 15^2} = 5\sqrt{1^2 + 3^2} = 5\sqrt{1+9} = 5\sqrt{10}$.

20. What is the shortest distance between the point $(5, -3)$ and the line $y = -\frac{1}{7}x - 8$?

The line can be rewritten $7y + x + 56 = 0$, so that the distance will be $\frac{|7(-3)+5+56|}{\sqrt{7^2+1^2}} = \frac{|-21+61|}{\sqrt{49+1}} = \frac{40}{\sqrt{50}} = \frac{40}{5\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$.

21. The point $(-9, 0)$ is reflected across the line $y = x + 1$ to point L, which is then reflected across the line $x = -7$ to point M. What are the coordinates, in the form (x, y), of point M?

Drawing a square of side 8 with $y = x + 1$ as a diagonal, you can see that point L is at $(-1, -8)$, so that point M will be at $(-13, -8)$.

22. What are the coordinates, in the form (x, y), of the vertex of the parabola $y = 2x^2 - 8x + 31$?

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{-8}{2\times 2} = \frac{8}{4} = 2$, so the corresponding y-value is $y = 2 \times 2^2 - 8 \times 2 + 31 = 2 \times 4 - 16 + 31 = 8 + 15 = 23$, for an answer of (2,23).

23. When a positive integer's digits are reversed to form a new positive integer that does not begin with zero, the resulting number is 693 less than the original number. When all numbers satisfying this condition are written in ascending order, what number is 11th in this list?

Obviously, the number in question cannot be a two-digit number. When a three-digit number of the form ABC is reversed to CBA, the difference is $99(A - C)$, so for this problem $A - C = \frac{693}{99} = 7$ could work. The smallest numbers with this property are 801, 811, 821, 831, 841, 851, 861, 871, 881, and 891 (ten so far), then 902.

24. A troop of Girl Scouts contributes equally to buy a kayak. If there had been one more girl in the troop, each girl's contribution would have been five dollars less. If there had been another girl beyond that one, each girl's contribution would have been another four dollars less. How much did the kayak cost?

We can write $K = Gc = (G+1)(c-5) = (G+2)(c-9)$. The latter parts can be expanded and rearranged to give $5 = c - 5G$ and $18 = 2c - 9G$. When twice the former is subtracted from the latter, we get $8 = G$, so that $c = 45$ and $K = 360$.

25. If Q quarters can exactly buy R ounces of fudge, how many ounces of fudge could be purchased with D dimes?

The price of fudge is $\frac{25Q}{R}$ cents per ounce, so we can buy $\frac{10D}{\frac{25Q}{R}} = \frac{10DR}{25Q} = \frac{2DR}{5Q}$ ounces.

26. If 8 Nether Engines are equivalent to 12 Oranges, 9 Picks are equivalent to 15 Quills, and 10 Nether Engines are equivalent to 18 Picks, how many Quills are equivalent to 900 Oranges?

Going through Nether Engines, then Picks, to Quills, 900 Oranges are equivalent to $900 \times \frac{8}{12} \times \frac{18}{10} \times \frac{15}{9} = 900 \times \frac{2}{3} \times \frac{9}{5} \times \frac{5}{3} = 100 \times 2 \times 9 = 1800$ Quills.

27. I have 60 coins in my piggy bank, with each being either a nickel, a dime, or a quarter. The number of dimes is one more than twice the number of nickels, and the total value of the coins is \$10.85. How many quarters are there?

We can write $n + d + q = 60$, $d = 2n + 1$, and $5n + 10d + 25q = 1085$. The last equation can be reduced to $n + 2d + 5q = 217$. Using the second equation to replace d, we get $3n + q = 59$ and $5n + 5q = 215$, which becomes $n + q = 43$. The former minus the latter gives $2n = 16$, so that $n = 8$, giving $q = 35$.

28. What value(s) of v satisfy $\frac{v+1}{2v-3} = \frac{3+4v}{5-6v}$?

Cross-multiplying gives $-6v^2 - v + 5 = 8v^2 - 6v - 9$, which becomes $0 = 14v^2 - 5v - 14$. The Quadratic Formula gives $v = \frac{5 \pm \sqrt{5^2+4\times14\times14}}{2\times14} = \frac{5\pm\sqrt{25+784}}{28} = \frac{5\pm\sqrt{809}}{28}$.

29. If $w(a) = 2a(3a+4)^5$, evaluate $w(-2)$.

$w(-2) = 2(-2)(3(-2)+4)^5 = -4(-2)^5 = -4(-32) = 128$

30. What ordered triple(s), in the form (h, j, k), satisfy the equations $h + j + k = 5$, $2h + j - k = 12$, and $h - j + k = -9$?

Subtracting the last equation from the first gives $2j = 14$, so $j = 7$, so that the first two equations are $h + k = -2$ and $2h - k = 5$. Adding these gives $3h = 3$, so $h = 1$ and $k = -3$, for an answer of $(1,7,-3)$.

31. When Mr. Plough writes a quadratic equation of the form $m^2 + Nm + P = 0$ on the board, Carmen miscopies the value of N, getting roots of -2 and 10. Rowan miscopies the value of P, getting roots of -7 and 8. What are the correct roots of Mr. Plough's equation?

Carmen's roots must have the correct product of $-2 \times 10 = -20$. Rowan's sum of $-7 + 8 = 1$ must be correct as well, so the real roots have a sum of 1 and a product of -20, and must be -4 and 5.

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32. A rectangular picture's width is twice its height, and the picture is in a rectangular frame that extends 2 cm on all sides of the picture. If the area of only the picture frame (not including the area of the picture it contains) is 208 cm^2, what is the height of the picture in centimeters?

The picture is h by $2h$, so the frame's outer edge is $h+4$ by $2h+4$. The area of the frame is $(h+4)(2h+4)-h(2h)=2h^2+12h+16-2h^2=12h+16=208$, which becomes $12h=192$, giving $h=16$.

33. A right triangle has a hypotenuse measuring 12 cm and a leg measuring 9 cm. What is the length, in centimeters, of the other leg?

The Pythagorean Theorem gives $\sqrt{12^2-9^2}=3\sqrt{4^2-3^2}=3\sqrt{16-9}=3\sqrt{7}$.

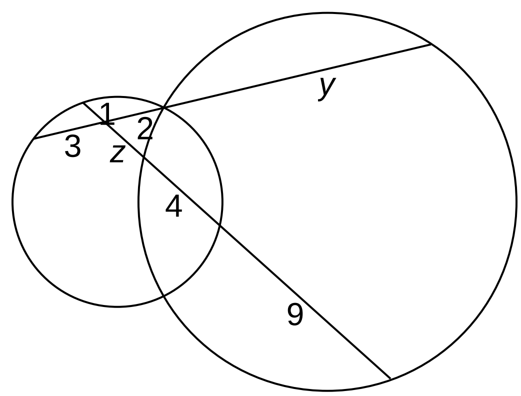
34. What is the area, in square meters, of an isosceles triangle with sides measuring 10 m and 4 m?

The triangle cannot be 4-4-10, so it must be 10-10-4. The altitude to the 4 can be determined to be $\sqrt{10^2-2^2}=\sqrt{100-4}=\sqrt{96}=4\sqrt{6}$, so that the area will be $\frac{1}{2}\times 4\times 4\sqrt{6}=8\sqrt{6}$.

35. What is the most specific name for a triangle with no two angles congruent?

In a triangle, the smallest side is opposite the smallest angle and the largest side is opposite the largest angle, so if the angles are not congruent, the sides cannot be, making this a scalene triangle.

36. In the figure to the upper right, all segment lengths are given in meters. Evaluate $y+z$.



Using the chords in the left circle, $2\times 3=1\times(z+4)$, so that $z+4=6$, giving $z=2$. Using the secants of the right circle, $2(2+y)=z(z+13)$, so that $4+2y=2\times 15=30$, giving$2y=26$ and $y=13$. $y+z=13+2=15$.

37. In a particular quadrilateral, opposing sides are congruent to one another (and there are two such pairs). What name can be given to *any* quadrilateral with this property?

This property is sufficient to make the quadrilateral a parallelogram.

38. What is the perimeter, in meters, of a $150°$ sector of a circle with a diameter of 18 m?

The perimeter will have two radii of 9, and an arc measuring $\frac{150}{360}\times 18\pi=\frac{15\pi}{2}$, for an answer of $18+\frac{15\pi}{2}$.

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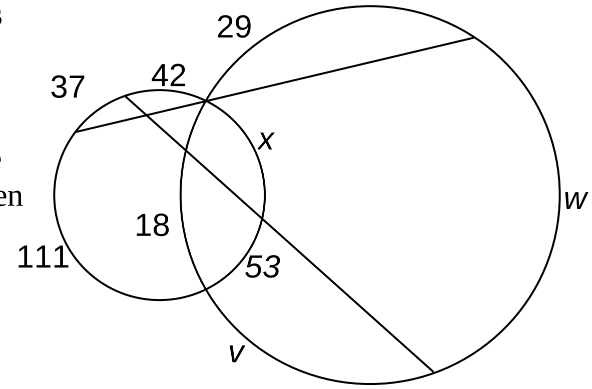
39. In the figure to the right, all given arc measures are in degrees. Evaluate $w + x$.



The left circle gives $x = 360 - 111 - 53 - 42 - 37 = 360 - 90 - 153 = 360 - 243 = 117$. The chords of the left circle let us find the angle between the secants of the right circle to be $\frac{117+37}{2} = \frac{154}{2} = 77$. The secants of the right circle let us write $\frac{w-117}{2} = 77$, so that $w - 117 = 154$ and thus $w = 271$, making our answer $w + x = 271 + 117 = 388$.

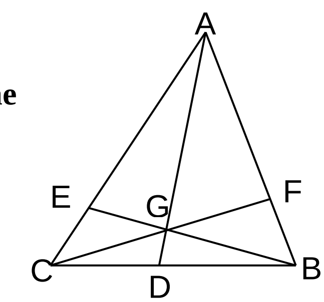
40. What is the area, in square meters, of an equilateral triangle that is circumscribed about a circle with a radius of 12 m?

Drawing the three altitudes creates six sub-triangles, each of which is 30-60-90, and each of which has a short side of 12. This makes the sides of the equilateral triangle $2 \times 12\sqrt{3} = 24\sqrt{3}$, for an area of $\frac{(24\sqrt{3})^2\sqrt{3}}{4} = (12\sqrt{3})^2\sqrt{3} = 144 \times 3\sqrt{3} = 432\sqrt{3}$.

41. What is the name of a polygon with eight sides?

This is just something you memorize: octagon.

42. In ΔABC to the right, cevians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at G. If the ratio between the areas of ΔACF and ΔBCF is 3 and the ratio between the areas of ΔACD and ΔABD is $\frac{3}{2}$, what is the ratio (expressed as a fraction if it's not an integer) of the area of ΔAEB to that of ΔCEB?



Ceva's Theorem discusses relationships between the ratios of side lengths in this type of triangle, which are equal to the ratios of sub-triangle areas. In this triangle, we can write $\frac{3}{1} \times \frac{2}{3} \times \frac{A\Delta CEB}{A\Delta AEB} = 1$, so that $\frac{A\Delta CEB}{A\Delta AEB} = \frac{1}{2}$ and our answer is 2.

43. A polyhedron has a surface area of 100 m^2 and a volume of 50 m^3. What is the volume, in cubic meters, of a similar polyhedron has a surface area of 2500 m^2?

The ratio of surface areas is $\frac{2500}{100} = 25$, so the ratio of distances is $\sqrt{25} = 5$, giving a volume ratio of $5^3 = 125$, so that the volume of the larger polyhedron is $50 \times 125 = 6250$.

44. What is the volume, in cubic meters, of a regular octahedron with edges measuring 8 m?

This is two right square pyramids joined at their bases. The square bases have areas of $8 \times 8 = 64$ and have heights of $\frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2}$, for a volume of $\frac{2}{3} \times 64 \times 4\sqrt{2} = \frac{512\sqrt{2}}{3}$.

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45. Two spherical balls are rolled into the same corner of a rectangular room with vertical walls, so that each touches both walls, the floor, and the other ball. What is the ratio of the radii of the larger ball to that of the smaller?

The center of the larger ball (with radius R) is $R\sqrt{3}$ from the corner, but this distance can also be described as $R + r + r\sqrt{3}$. Setting these equal to each other gives $R\left(\sqrt{3} - 1\right) = r\left(\sqrt{3} + 1\right)$, so that $\frac{R}{r} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{3+1+2\sqrt{3}}{3-1} = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}$.

46. What is the length, in meters, of the altitude to the longest side of a triangle with sides measuring 12 m, 15 m, and 21 m?

The area of this triangle will be $3^2 = 9$ times the area of a 4-5-7 triangle, which is $A = \sqrt{8 \times 1 \times 3 \times 4} = 4\sqrt{6}$, so the area of our triangle is $9 \times 4\sqrt{6} = 36\sqrt{6}$. The altitude to the 21 side is thus $\frac{2\times 36\sqrt{6}}{21} = \frac{2\times 12\sqrt{6}}{7} = \frac{24\sqrt{6}}{7}$.

47. A horse is tethered to outside of the $60°$ angle of a stable in the shape of a 30-60-90 triangle. If the hypotenuse of the stable measures 20 m and the horse's tether is 22 meters long, what is the area, in square meters, the horse can graze? Note: the horse cannot get into the stable.

$$A = \frac{5}{6}\pi \times 22^2 + \frac{5}{12}\pi \times 2^2 + \frac{1}{4}\pi \times 12^2 = \frac{5}{3}\pi \times 242 + \frac{5}{3}\pi + \pi \times 36$$
$$= \frac{\pi(1210 + 5 + 108)}{3} = \frac{1323\pi}{3} = 441\pi$$

48. What is the largest number of regions into which a line, a circle, and a triangle can divide a plane?

The circle and triangle can divide a plane into 8 regions. A line can go through four of those regions, so it will add four new regions for a total of $8 + 4 = 12$.

49. The complement of $\angle B$ is equal to the supplement of $\angle C$, while the supplement of $\angle B$ is $52°$ larger than the complement of $\angle D$. What is the positive difference, in degrees, between the supplements of $\angle C$ and $\angle D$?

We can write $90 - B = 180 - C$, which becomes $C = B + 90$. We can also write $180 - B = 90 - D + 52$, which becomes $B = D + 38$, so that $C = D + 38 + 90 = D + 128$. The answer we seek is $180 - D - (180 - C) = C - D = 128$.

50. Simplify in terms of $i = \sqrt{-1}$: $i(2 + 3i) - 4(5 - 6i) + (2i)^7$

$= 2i - 3 - 20 + 24i - 128i = -23 - 102i$

51. What are the coordinates, in the form (x, y), of the center of the hyperbola with equation $3x^2 - 5y^2 + 18x - 40y = 100$?

Completing the squares gives $3(x+3)^2 - 5(y+4)^2 \ldots$, so the center is $(-3, -4)$.

52. What is the maximum number of points in which equations of the forms $y = AB^{Cx} + D$ and $y = Fx^3 + Gx^2 + Hx + J$ can intersect?

The first equation is exponential, which means it consists of a long flat section, a curve, and a steep section. The second equation is cubic, which means it can reverse direction twice between three potentially steep sections. If the flat section of the exponential intersect the three steep sections of the cubic, the steep section of the exponential can eventually cross a steep section of the cubic, for a fourth intersection.

53. What value(s) of k satisfy $4^{k+1} - 5 \times 2^{k+5} + 1024 = 0$?

Expressing in terms of 2^k gives $4 \times 2^{2k} - 160 \times 2^k + 2^{10} = 0$. This is a quadratic in terms of 2^k, and can be factored to $(4 \times 2^k - 2^5)(2^k - 2^5) = 0$, with roots of 3 and 5.

54. How many distinguishable functions map the set of one-digit prime numbers onto the set of positive two-digit palindromic integers?

There are four one-digit primes (2, 3, 5, 7) and nine two-digit palindromes (11-99). Each of the four could be mapped to any of the nine, for an answer of $9^4 = 9 \times 729 = 6561$.

55. Suppose that m is directly proportional to $\frac{n^2}{p^3}$. If $m = 216$ when $n = p = 216$, what value of m corresponds to $n = 108$ and $p = 72$?

n was cut in half and *p* was divided by three, so $m = \frac{216 \times 3^3}{2^2} = 54 \times 27 = 729 \times 2 = 1458$.

56. \$1000.00 is invested at 2% annual interest compounded annually. If it's left alone, how much money, in dollars rounded to the nearest hundredth (i.e. cent), will be in the account after three years?

In the first year, \$20 is earned, for a total of \$1020. In the second year, \$20 and \$0.40 is earned, for a total of \$1040.40. In the third year, \$20, \$0.80, and \$0.008 are earned, for a total of \$1061.208, which rounds to \$1061.21.

57. What is the product of the roots of $2q^6 + 3q^4 + 5q^3 - 7q^2 - 8 = 0$?

The product of the roots is $(-1)^n \frac{z}{a} = (-1)^6 \left(-\frac{8}{2}\right) = -4$.

58. What is the sum of the seven complex seventh-roots of $7 - i$?

The roots in question satisfy $z^7 = 7 - i$, which can be written $z^7 - (7 - i) = 0$, which is a polynomial with many coefficients of 0, and specifically "*b*" is 0, so the sum of the roots is 0.

59. Express the base-11 number 754_{11} as a base-10 number.

$7 \times 11^2 + 5 \times 11 + 4 = 7 \times 121 + 55 + 4 = 847 + 59 = 906$

60. Express the base-9 number 1358_9 as a base-3 number.

A digit that represents 9^n can be represented as a combination of 3^{2n} and 3^{2n+1}, so each digit of the base-9 number can be represented by two digits of a base-3 number, giving 1101222_3.

61. What is the sum of the positive integer factors of 240?

$240 = 2^4 \times 3 \times 5$, so the sum of its factors will be $(1 + 2 + 4 + 8 + 16)(1 + 3)(1 + 5) = 31 \times 4 \times 6 = 124 \times 6 = 744$.

62. How many positive integer factors of 6000 are multiples of 6?

$6000 = 2^4 \times 3 \times 5^3$, and $6 = 2 \times 3$, so the numbers in question could have from one to four 2's (four choices), must have one 3 (one choice), and could have from zero to three 5's (four choices), for a total of $4 \times 1 \times 4 = 16$.

63. Tammy writes the number 123 and every 235th number after that, while Ula writes the number 347 and every 450th number after that. What is the smallest difference between two of the numbers written?

The initial numbers are $347 - 123 = 224$ apart, and the deltas of $235 = 5 \times 47$ and $450 = 2 \times 3^2 \times 5^2$ have a GCF of 5, which means that over an infinite number of deltas, the initial numbers can change their separation by any number of 5's, so that at some point there will be two numbers that are only 1 apart (subtracting 5's eventually gives -1).

64. What is the sum of the twenty smallest palindromic integers greater than 9000?

There are ten palindromes of the form 9xx9 from 9009 to 9999, which form five pairs with sums of 19008, for a subtotal of 95040. The next ten palindromes are of the form 10y01 from 10001 to 10901, which form five pairs with sums of 20902, for a subtotal of 104510 and a grand total of 199550.

65. What is the largest prime number that is a factor of 6,531,840?

Clearly 8 is a factor of this number (last three digits), as is 9 (digital sum), as is 5, so $6531840 = 360 \times 18144$. Clearly 8 is a factor of 18144, as is 9, so $6531840 = 360 \times 72 \times 252$. Clearly 4 is a factor of 252, as is 9, so $6531840 = 360 \times 72 \times 36 \times 7$, making 7 the largest prime factor.

66. What is the sum of the terms of an infinite geometric sequence with first term 24 and common ratio $\frac{1}{2}$?

$$S = \frac{a}{1-r} = \frac{24}{1-\frac{1}{2}} = \frac{24}{\frac{1}{2}} = 48$$

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67. What is the third term of a harmonic sequence with first term 4 and second term 6?

The terms of a harmonic sequence are the reciprocals of the terms of an arithmetic sequence, which in this case begins $\frac{1}{4}, \frac{1}{6}, \ldots$ The next term will be $2 \times \frac{1}{6} - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, making the next term of the harmonic sequence 12.

68. What is the sum of the three missing terms of the sequence beginning 1, 3, 4, 6, 10, 14, 15, 21, 24, __, __, __, 45, 55, 44, …?

Perhaps the evenly-spaced 4, 14, 24, __, 44 made you think this could be an interspersed sequence, which it is. The remaining terms are 1, 3, 6, 10, 15, 21, __, __, 45, 55, …, which is the triangular numbers. The answer is thus $34 + 28 + 36 = 98$.

69. What is the 31st term of an arithmetic sequence with first term 32 and common difference 33?

$32 + 30 \times 33 = 32 + 990 = 1022$

70. What is the missing term of the sequence WON, TOO, FOR, __, ATE?

These are homophones for the numbers one, two, four, and eight. The missing number is six, which is a homophone for sics (or sicks), which means attacks.

71. What is the sum of the 20 smallest positive perfect squares?

$$\frac{20 \times 21 \times 41}{6} = 10 \times 7 \times 41 = 10 \times 287 = 2870$$

72. What is the sum of the two-digit positive integers that are not divisible by 7?

The sum of ALL the two-digit numbers is $45 \times 109 = 4905$. The sum of the multiples of 7 (14 to 98) is $\frac{13 \times 112}{2} = 13 \times 56 = 728$, and the difference is 4177.

73. When three cards are drawn from a standard 52-card deck, what is the probability that at least two of the three are of the same rank? E.g. two Kings or three Sevens, etc.

This will be 1 minus the probability that the cards are of three different ranks, which is $1 - 1 \times \frac{48}{51} \times \frac{44}{50} = 1 - \frac{16}{17} \times \frac{22}{25} = 1 - \frac{352}{425} = \frac{73}{425}$.

74. When four fair six-sided dice are rolled, what is the probability that the sum of the numbers shown on their upper faces is 8?

There are $6^4 = 1296$ ways to roll four dice. The "good" ways are 1115 (four ways), 1124 (twelve ways), 1133 (six ways), 1223 (twelve ways), and 2222 (one way), for a probability of $\frac{35}{1296}$.

75. A trusted friend draws two marbles from a bag containing three red, four white, and five blue marbles, and tells you that they are not both blue (either one might be blue, however). What is the probability that the two marbles are the same color?

There are $12c2 = 66$ ways to draw two marbles, but we didn't do it the $5c2 = 10$ ways that produced two blues, so there are really only $66 - 10 = 56$ ways we might have drawn the marbles. Of those, there are $3c2 = 3$ ways we might have drawn two reds and $4c2 = 6$ ways we might have drawn two whites, for a probability of $\frac{3+6}{56} = \frac{9}{56}$.

76. In how many distinguishable ways can the letters in the word "CIRCUMFERENCE" be arranged?

There are 13 letters, with 3 C's, 2 R's, and 3 E's, for a total of $\frac{13!}{3!\times 2!\times 3!} = 13 \times 11 \times 10 \times 9 \times 8 \times 7 \times 5 \times 4 \times 3 \times 2 = 1001 \times 720 \times 120 = 1001 \times 6 \times 14400 = 1001 \times 86400 = 86{,}486{,}400$ arrangements.

77. The members of the art club were surveyed on gender (boy or girl) and favorite color (red, blue, or green), with the following results: the number of members who liked red was half the number of boys, the number of members who liked green was five times the number of boys who liked blue, the number of girls was equal to the number of boys who liked green, and the number of members who liked blue was half the number of boys who liked red. If exactly one girl liked blue, what is the fewest number of members that could be in the art club?

Let the number of reds be *z*, so that the number of boys is *2z*. Similarly, let the number of boys who like blue be *y*, so that the number of greens is *5y*. Also, let the number of girls and the number of boys who like green be *x*. Finally, let the number of blues be *w*, so that the number of boys who like red is *2w*. The first thing that jumps out is that $w = y + 1$. In addition, $2w + x + y = 2z$ and $2z + x = z + 5y + w$. This is three equations in four unknowns, but since we're just looking for a minimum (integer) solution, the fact that we're not fully constrained is fine. Using the first equation to substitute for *w* in the others and rearranging, we get $3y + x - 2z = -2$ and $-6y + x + z = 1$. Subtracting the second from the first gives $9y - 3z = -3$, which simplifies to $3y - z = -1$, so that $z = 3y + 1$. Adding the first to twice the second gives $-9y + 3x = 0$, so that $x = 3y$. $y = 0$ would mean $w = 1$, $z = 1$, and $x = 0$, but this would mean that there were 2 boys who liked red but only 1 person overall who liked red. $y = 1$ would mean $w = 2$, $z = 4$, and $x = 3$, which works and gives a total of $2 \times 4 + 3 = 8 + 3 = 11$ members.

78. In the hot new casino game, you pay twenty dollars for a chance to flip four fair coins, after which the Floupier will count the number of tails (T) and pay you 3^T dollars. What is your expected gain if you play this game? (If you are expected to lose money, your answer would be negative.)

After you pay your \$20, you have a $\frac{1}{16}$ chance of getting \$1 back, $\frac{4}{16}$ of \$3, $\frac{6}{16}$ of \$9, $\frac{4}{16}$ of \$27, and $\frac{1}{16}$ of \$81, for a total of $\frac{1+12+54+108+81}{16} = \frac{256}{16} = 16$ and an expected gain of $16 - 20 = -4$.

79. In one segment of the hit game show "That's the Price, Right?", host Honty Mall asks the contestant to point at one of five suitcases. Two suitcases are known to contain a Good Prize (and Honty knows which they are), while the others all contain Nothingness. After the contestant points, Honty opens a suitcase that he knows has Nothingness and shows it to the contestant, then offers the contestant the opportunity to change which suitcase she is pointing to. If the contestant points to a new suitcase that is neither her first choice nor the one Honty opened, what is the probability that it contains a Good Prize?

When I initially choose a suitcase, it contains $\frac{2}{5}$ of a Good Prize, leaving $\frac{10}{5} - \frac{2}{5} = \frac{8}{5}$ spread among the other suitcases. Seeing the revealed Nothingness says nothing about the suitcase I've chosen, so I still have $\frac{2}{5}$ of a Good Prize, and there are still $\frac{8}{5}$ among the other three suitcases, meaning each has $\frac{\frac{8}{5}}{3} = \frac{8}{15}$ of a Good Prize.

80. I draw two cards from a standard 52-card deck, look at them, and truthfully tell you that they are not of the same suit. What is the probability that I drew a pair (two cards of the same rank)?

There are $52c2 = 26 \times 51$ ways to choose two cards, but we know I didn't do it in the $4 \times 13c2 = 4 \times 13 \times 6$ ways that would have matched suits. Also, there are $13 \times 4c2 = 13 \times 2 \times 3$ ways to get a pair (none of which involved matching suits), so our probability is $\frac{13\times2\times3}{26\times51-4\times13\times6} = \frac{2\times3}{2\times51-4\times6} = \frac{3}{51-2\times6} = \frac{1}{17-2\times2} = \frac{1}{13}$.

81. If $\boldsymbol{U} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$, what is the sum of the elements of $\boldsymbol{U^{-1}}$?

The inverse matrix is the transposed matrix of cofactors, divided by the determinant of the original matrix, so $U^{-1} = \frac{1}{14}\begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$, and the sum of its elements is $\frac{6}{14} = \frac{3}{7}$.

82. What is the volume of the tetrahedron with vertices at the points $(\mathbf{1}, \mathbf{1}, \mathbf{1})$, $(\mathbf{-2}, \mathbf{3}, \mathbf{-4})$, $(\mathbf{0}, \mathbf{1}, \mathbf{2})$, and $(\mathbf{4}, \mathbf{3}, \mathbf{-2})$?

If we subtract points to get vectors from the point $(1,1,1)$ along the adjacent edges, we get $< -3,2,-5 >$, $< -1,0,1 >$, and $< 3,2,-3 >$. The cross product of any two of those will produce a perpendicular vector whose magnitude is $m = ab\sin\theta$, twice the area of the enclosed face. The dot product of that vector with the remaining original vector will be $cm\cos\phi$, which will be six times the volume of the tetrahedron. These operations can be done simultaneously through the determinant $\frac{1}{6}\begin{vmatrix} -3 & 2 & -5 \\ -1 & 0 & 1 \\ 3 & 2 & -3 \end{vmatrix} = \frac{1}{6}\Big(-(-1)\big(-6-(-10)\big) + (-1)(-6-6)\Big) = \frac{1}{6}(4+12) = \frac{16}{6} = \frac{8}{3}$.

83. What is the equation, in the form $Ax + By + Cz = -41$ of the plane through the points $(1,2,3)$, $(2,-3,5)$, and $(-2,4,1)$?

Subtracting to get two vectors in the plane gives $< 1,-5,2 >$ and $< -3,2,-2 >$. Crossing these gives a vector perpendicular to the plane: $< 6,-4,-13 >$. These should be the coefficients in the equation of the plane: $6x - 4y - 13z = d$. Substituting the first point gives $d = 6 \times 1 - 4 \times 2 - 13 \times 3 = 6 - 8 - 39 = -41$, which means our answer is $6x - 4y - 13z = -41$.

84. What is the mean of the mode, median, and range of the dataset $\{1,4,8,13,19,17,14,10,5,8,12\}$?

The mode is 8, the range is $19 - 1 = 18$, and the median is 10, for an answer of $\frac{8+18+10}{3} = \frac{36}{3} = 12$.

85. In a seven-element set of integer test scores from 0 to 100 inclusive, the mean is 69, the unique mode is 74, and the median is 70. What is the maximum possible value of the range?

The set in ascending order is __, __, __, 70, 74, 74, __. Having a large range will mean having a large largest number and a small smallest number. Having a mean of 69 means that the sum of the seven elements must be $7 \times 69 = 490 - 7 = 483$. We've already got elements that sum to 218, so the remaining elements must sum to $483 - 218 = 165$. If one were 100 and another were 0, we could use 30 and 35 for the others and have the maximum possible range of 100.

86. Set V is the set of all positive three-digit integers that contain a 7. Set W is the set of all positive multiples of 12 less than 1000. How many elements are in the set $W \cup V$?

Set W has $\left\lfloor\frac{1000}{12}\right\rfloor = 83$ elements. Set V has 100 elements that start with 7, 90 with 7 in the middle, and 90 with 7 at the end. However, this double-counts the 10 that start with 77, the 9 that end with 77, and the 10 that are 7x7, so these should be subtracted from our count. However, we triple-counted, then triple-subtracted, the number 777, so we should add 1 to our count, for a total of $100 + 90 + 90 - 10 - 10 - 9 + 1 = 280 - 28 = 252$ elements in Set V. Our answer should be $83 + 252 = 335$, except that some elements of W are also in V. There are no multiples of 12 that end in 7, but there are six multiples of 12 of the form 27x, 37x, 57x, 67x, 87x, and 97x. In addition, there are $\left\lfloor\frac{800}{12}\right\rfloor - \left\lfloor\frac{700}{12}\right\rfloor = 66 - 58 = 8$ that start with 7, for an answer of $335 - 6 - 8 = 321$.

87. In the cryptarithm below, in which each instance of a letter represents the same digit and no two different letters represent the same digit (e.g. if one A is a 1, every A is a 1 and no B is a 1), what is the largest possible value of the four-digit number $ABCD$?

$$\begin{array}{r} BCD \\ \underline{+ABC} \\ DAB \end{array}$$

Obviously, A cannot be 9. If A were 8, B would be 1 and D would be 9, but then there is no possible value for C. If A were 7, B could be 1 or 2 and D could be 8 or 9, but again there is no possible value for C. If A were 6, B could be 1, 2, or 3, and D could be 7, 8, or 9, in which case C could be 4 if B were 1 and D were 7, for an answer of 6147.

88. The numerals 1, 2, 3, 5, and 7 and the operators $+, -, \times,$ and $\div$ are used exactly once each, along with any number of parentheses, to create an expression. What is the maximum possible value when such an expression is evaluated?

Our chances to get really big would involve multiplying by a big number or dividing by a small fraction. We may only divide once, so we cannot create a small fraction and then divide by it, so we should focus on multiplying, using the larger numbers. Subtracting and dividing will decrease our number, so we should try to use the smaller numbers for this which works well with our multiplication strategy. If we use 7, 5, and 3 together for the addition and multiplication, we would like to multiply two numbers that are close to one another (given the fixed sum), which would be $7 \times (5 + 3) = 7 \times 8 = 56$. Now we need to divide and subtract using 1 and 2. The best way is to divide first to create the fraction $\frac{1}{2}$, then subtract this, getting $56 - \frac{1}{2} = \frac{112-1}{2} = \frac{111}{2}$.

89. When five people sit on a bench, Victor sits somewhere to Willa's left, Xavier sits somewhere to Yessica's right, and Zane doesn't sit next to Willa. How many different seating arrangements are possible?

The first two groups are seated V?W and Y?X, which can be arranged in $4c2 = 6$ ways (pick any two spots for the first group; there is only one way for them to sit in them, after which there is only one way for the other group to sit). However they seat themselves, there are five places that Z might sit (either end, or the three central seats), but two of them are next to W, so really there are only three choices, for an answer of $3 \times 6 = 18$.

90. In ΔFGH, $FG = 36$ m, $m\angle F = 75°$, and $m\angle G = 30°$, what is the length, in meters, of $\overline{GH}$?

This turns out to be a 30-75-75 triangle, so it's isosceles! That means that $GH = 36$.

91. Evaluate: $\sec\left(\frac{119\pi}{6}\right)$

$\frac{120\pi}{6}$ would be equivalent to 0, so $\frac{119\pi}{6}$ is equivalent to $-\frac{\pi}{6}$, which has a cosine of $\frac{\sqrt{3}}{2}$, so that its secant is $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

92. What is the area, in square meters, of ΔJKL if $JK = 18$ m, $KL = 36$ m, and $m\angle K = 120°$?

$$A = \frac{1}{2}ab\sin C = \frac{1}{2} \times 18 \times 36 \times \frac{\sqrt{3}}{2} = 18 \times 9\sqrt{3} = 162\sqrt{3}$$

93. What are the coordinates, in rectangular (x, y, z) form, of the point with spherical coordinates $\left(12, \frac{\pi}{3}, \frac{5\pi}{4}\right)$, where the second angle is the azimuthal angle?

Converting from ρ and ϕ, we get $z = \rho\cos\phi = 12\cos\frac{\pi}{3} = 12 \times \frac{1}{2} = 6$ and $r = \rho\sin\phi = 12\sin\frac{\pi}{3} = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3}$. Converting from r and θ, we get $y = r\sin\theta = 6\sqrt{3}\sin\frac{5\pi}{4} = 6\sqrt{3} \times \left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{6}$, with x working similarly for an answer of $(-3\sqrt{6}, -3\sqrt{6}, 6)$.

94. What is the name for the shape defined by the parametric equations $x = t^2 - 5$ and $y = 2t^2 + 1$ for real values of t?

Rearranging the first equation gives $t^2 = x + 5$, making the second equation $y = 2(x + 5) + 1 = 2x + 10 + 1 = 2x + 11$, so the shape is part of a line. Because $t^2 \geq 0$, $x \geq -5$, so instead of a line, we have a ray.

95. If $m(n) = 2n(n^2 + 1)^3$, evaluate $m'(1)$.

$m'(n) = 2(n^2 + 1)^3 + 2n \times 3(n^2 + 1)^2 \times 2n$, so $m'(1) = 2(1^2 + 1)^3 + 2 \times 3(1^2 + 1)^2 \times 2 = 2 \times 2^3 + 2 \times 3 \times 2^2 \times 2 = 16 + 48 = 64$.

96. A 25-foot ladder is standing on horizontal ground and propped against a vertical wall, but it's slipping so that its top is sliding down the wall at 2 ft/sec. At how many feet per second is the base of the ladder sliding away from the wall at the moment when the top of the ladder is only 7 feet off the ground?

If the height is y and the distance from the wall is x, $x^2+y^2=25^2$ will always be true. Because of this, we can implicitly differentiate to get $2xx'+2yy'=0$, so that $xx'=-yy'$ at all times. At the moment in question, $x^2+7^2=25^2$, so $x^2=625-49=576$ and $x=24$. In addition, $24x'=-7(-2)$ gives $x'=\frac{14}{24}=\frac{7}{12}$.

97. What is the total area enclosed by the functions $y=x^2$ and $y=x^3$ and the lines $x=-2$ and $x=2$?

The functions intersect at $x=0$ and $x=1$, but only cross at $x=1$. Because of this, we need two integrals: $\int_{-2}^{1}(x^2-x^3)dx+\int_{1}^{2}(x^3-x^2)dx=\left(\frac{1}{3}x^3-\frac{1}{4}x^4\right)\Big|_{-2}^{1}+\left(\frac{1}{4}x^4-\frac{1}{3}x^3\right)\Big|_{1}^{2}=\left(\frac{1}{3}-\frac{1}{4}+\frac{8}{3}+4\right)+\left(4-\frac{8}{3}-\frac{1}{4}+\frac{1}{3}\right)=8-\frac{1}{2}+\frac{2}{3}=8+\frac{1}{6}=\frac{49}{6}$.

98. What is the equation of the line tangent to the graph of $y=3x^2-1$ at the point where $x=-2$?

When $x=-2$, $y=3(-2)^2-1=3\times4-1=12-1=11$. $\frac{dy}{dx}=6x=6(-2)=-12$, so the equation of the tangent line is $y=-12x+b$. Substituting gives $11=-12(-2)+b=24+b$, so that $b=-13$ for an answer of $y=-12x-13$.

99. What is the average value of the function $p(q)=2q(3q^2+4)$ on the interval $[0,16]$?

The average value is $\frac{\int_0^{16}2q(3q^2+4)dq}{16-0}=\frac{1}{16}\times\frac{1}{6}(3q^2+4)^2\Big|_0^{16}=\frac{1}{96}(772^2-4^2)=\frac{1}{6}(193^2-1)=\frac{37248}{6}=6208$.

100. A spherical amoeba's radius is increasing at a rate of 24m per second. At what rate, in square meters per second, is its surface area increasing at the moment it is 24m in diameter?

$SA=4\pi r^2$, so $SA'=8\pi rr'=8\pi\times12\times24=4\pi\times24^2=48^2\pi=2304\pi$