6 Applications of Integration

6.3 Volume: The Shell Method

Objectives

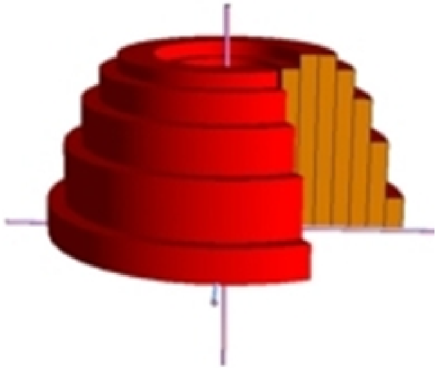
- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

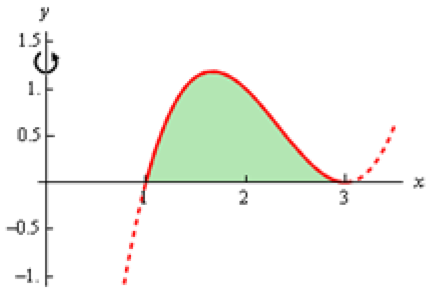
The Shell Method

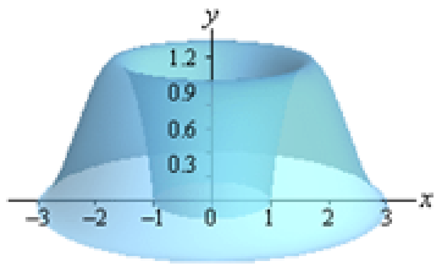
An alternative method for finding the volume of a solid of revolution is called the **shell method** because it uses cylindrical shells.

A comparison of the advantages of the disk and shell methods is given later in this section.

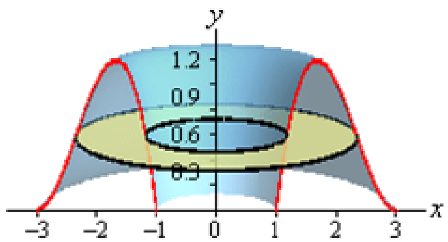


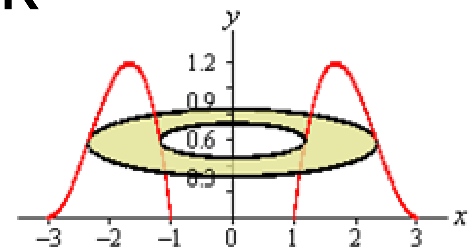
The Disk & Shell Methods



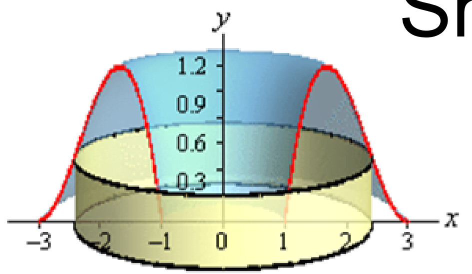


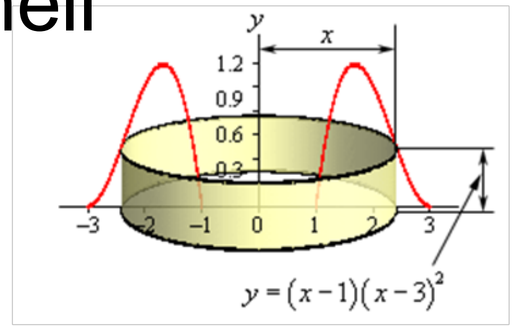
Disk





Shell





The Shell Method

Assume that the plane region in Figure 6.28 is revolved about a line to form the indicated solid.

If you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the *x*-axis, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

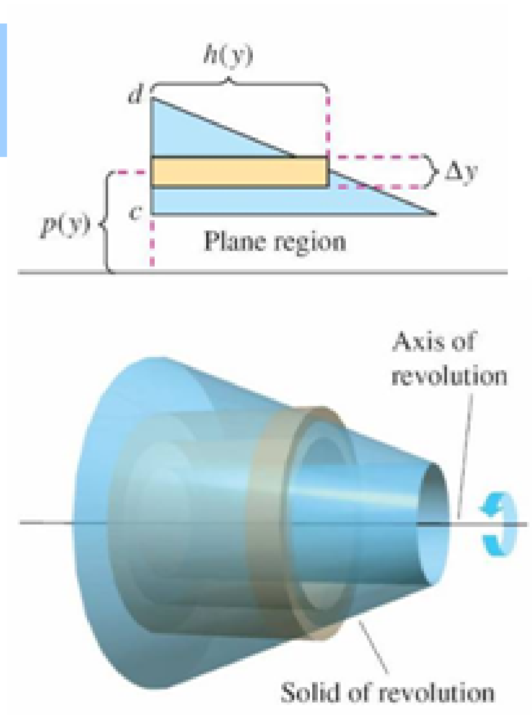


Figure 6.28

The Shell Method

You can approximate the volume of the solid by n such shells of thickness Δy, height $h(y_i)$, and average radius $p(y_i)$.

$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\,\Delta y = 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

This approximation appears to become better and better as $\|\Delta\| \to 0 \, (n \to \infty)$. So, the volume of the solid is

$$\begin{aligned}\text{Volume of solid} &= \lim_{\|\Delta\|\to 0} 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y \\ &= 2\pi\int_c^d [p(y)h(y)]\,dy.\end{aligned}$$

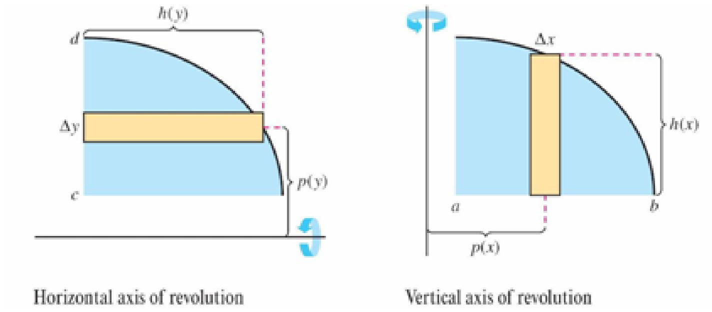
The Shell Method

THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 6.29

Horizontal Axis of Revolution	*Vertical Axis of Revolution*
$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\,dy$	$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\,dx$

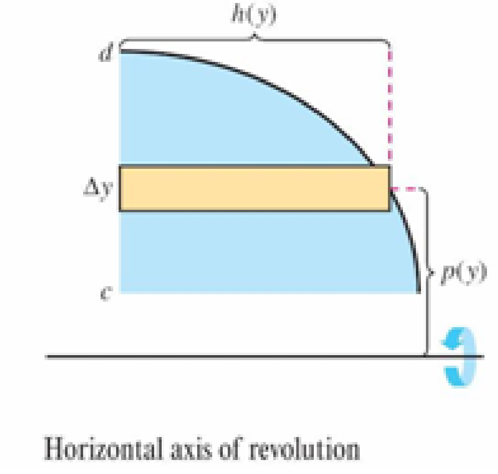
radius: $p(y) = y$
height: $h(y) = f(y)$

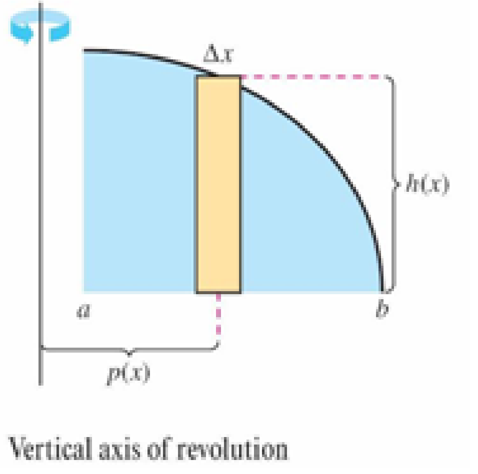


radius: $p(x) = x$
height: $h(x) = f(x)$

Figure 6.29

The Shell Method





radius: p(y) = y
height: h(y) = f(y)

radius: p(x) = x
height: h(x) = f(x)

Example 1 – *Using the Shell Method to Find Volume*

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \le x \le 1$) about the y-axis.

Solution on next pages

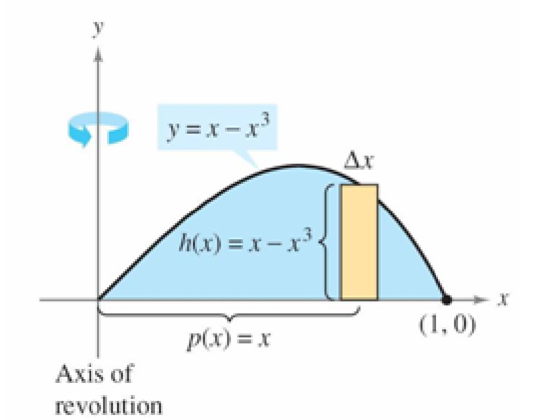


Figure 6.30

Example 1 – *Using the Shell Method to Find Volume*

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \leq x \leq 1$) about the y-axis.

Solution:

Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 6.30. The width Δx indicates that x is the variable of integration.

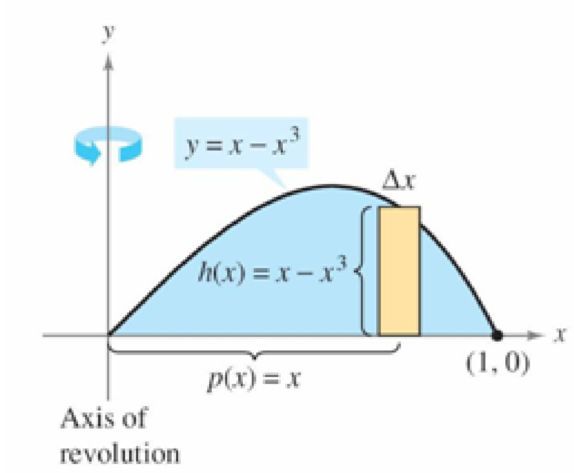


Figure 6.30

Example 1 – *Solution*

cont'd

The distance from the center of the rectangle to the axis of revolution is $p(x) = x$, and the height of the rectangle is $h(x) = x - x^3$.
Because x ranges from 0 to 1, the volume of the solid is

$$V = 2\pi \int_a^b p(x)h(x)\,dx = 2\pi \int_0^1 x(x - x^3)\,dx \qquad \text{Apply shell method.}$$

radius is x

$$= 2\pi \int_0^1 (-x^4 + x^2)\,dx \qquad \text{Simplify.}$$

$$= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 \qquad \text{Integrate.}$$

$$= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right) = \frac{4\pi}{15}.$$

Comparison of Disk and Shell Methods

Shell Method -vs- Disk Method

Shell Method: Use the shell method if the equation is solved for the same variable as the axis that you are rotating around.
For example: volume of the area bounded by **y** = $-3x^2+8x$ and y=0 and rotated around the **y-axis** would be easiest using the shell method. (The upper and lower **limits** for the integrand are **on the axis you did not rotate around**.)

Disk Method: Use the disk method if the equation is solved for the variable that is not the axis you are rotating around.
For example: volume of the area bounded by **y** = $-3x^2+8x$ and y=0 and rotated around the **x-axis** would be easiest using the disk method. (The upper and lower **limits** for the integrand are **on the axis you rotated around**.)

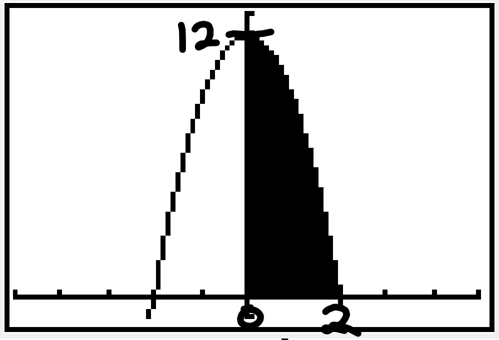
	Disk/Washer Method	Shell Method
Rotate about x-axis	$\int$ x's limits from x-axis	$\int$ y's limits from y-axis
Rotate about y-axis	$\int$ y's limits from y-axis	$\int$ x's limits from x-axis

Example 2: Find the volume of the area bounded by **y** = $-3x^2+12$, y=0, & x=0 rotated about the **y-axis.**

solved for x: $x = \sqrt{4 - \frac{1}{3}y}$

Disk Method:

$$\pi \int_0^{12} \sqrt{4-\frac{1}{3}y}^{\,2}\, dy = 24\pi$$



Shell Method:

$$2\pi \int_0^{2} x(-3x^2+12)\, dx = 24\pi$$

Find the volume of the area bounded by **y** = $-3x^2+12$, y=0 & x=0 rotated about the **x-axis.**

Disk Method:

$$\pi \int_0^{2} (-3x^2+12)^2\, dx = 153.6\pi$$

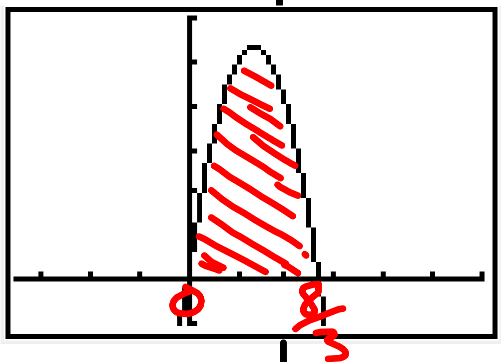
Shell Method:

$$2\pi \int_0^{12} y\sqrt{4-\frac{1}{3}y}\, dy = 153.6\pi$$

Example 3:
Find the volume of the area bounded by
y = $-3x^2+8x$ and y=0 rotated about the **y-axis.**

~~Disk Method~~:

can't solve for x, so we can't use this method.



Shell Method:

$$2\pi\int_0^{8/3} x(-3x^2+8x)\,dx \approx \underline{79.432}$$

Find the volume of the area bounded by
y = $-3x^2+8x$ and y=0 rotated about the **x-axis**.

Disk Method:

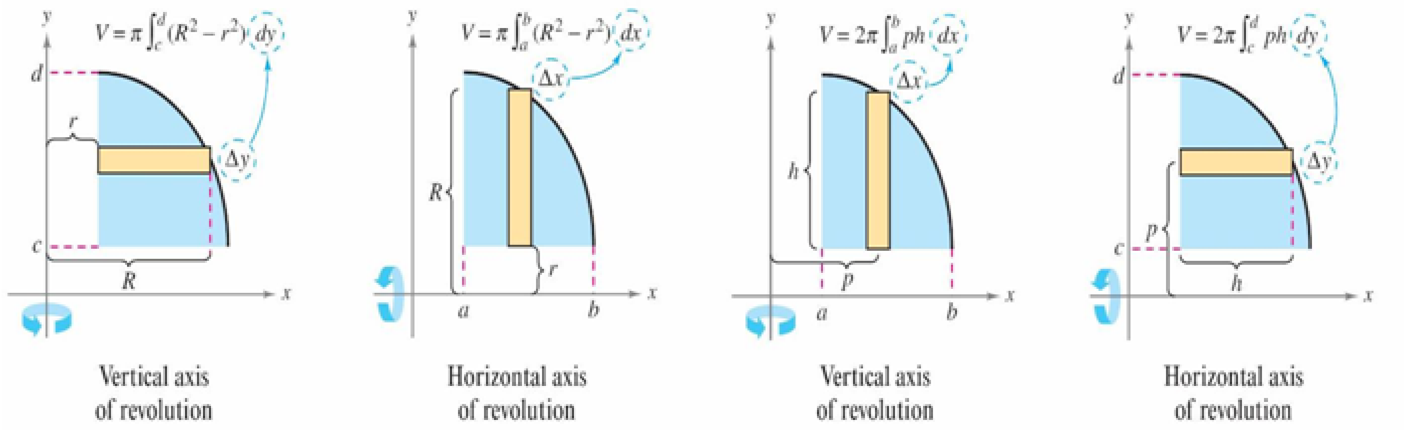
$$\pi\int_0^{8/3} (-3x^2+8x)^2\,dx \approx \underline{127.091}$$

~~Shell Method~~:

can't solve for x, so we can't use this method.

Comparison of Disk and Shell Methods

For the disk method, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution.



Disk method: Representative rectangle is perpendicular to the axis of revolution.

Shell method: Representative rectangle is parallel to the axis of revolution.

Figure 6.32

Example 4 – Which method is best?

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

The washer method requires two integrals to determine the volume of this solid. See Figure 6.33(a).

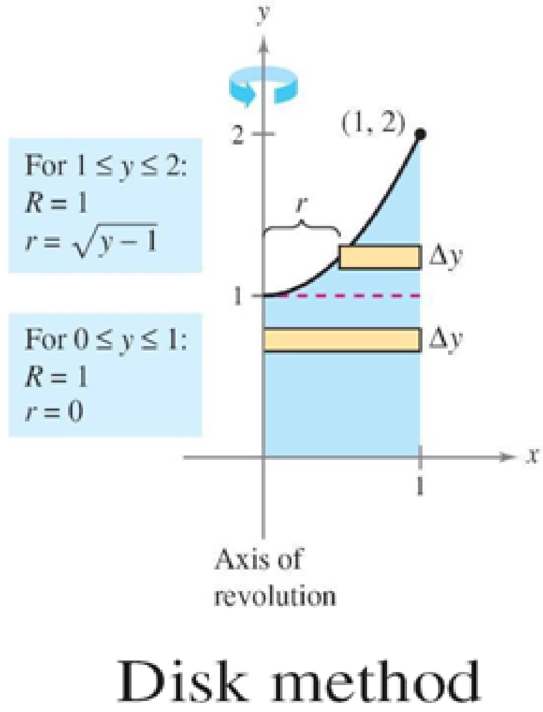


Figure 6.33(a)

Example 4 Solution: Disk/Washer Method

$$V = \pi \int_0^1 (1^2 - 0^2)\, dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy$$ Apply washer method.

$$= \pi \int_0^1 1\, dy + \pi \int_1^2 (2 - y)\, dy$$ Simplify.

$$= \pi\Big[y\Big]_0^1 + \pi\left[2y - \frac{y^2}{2}\right]_1^2$$ Integrate.

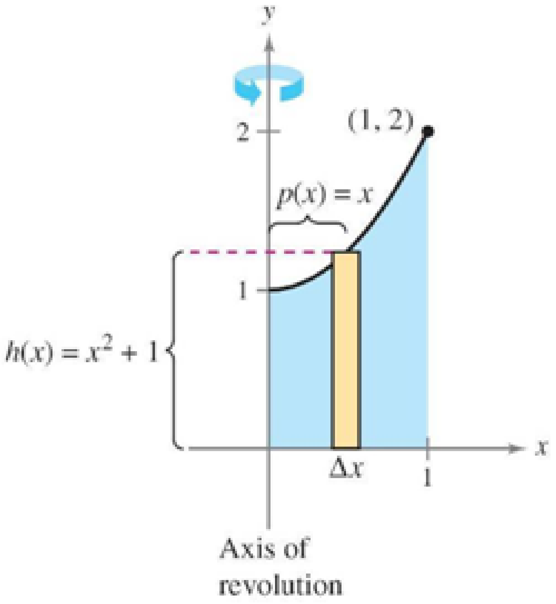
$$= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right)$$

$$= \frac{3\pi}{2}$$

Example 4: Use the Shell Method:

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Solution on next page



Example 4 *Shell Method Solution*

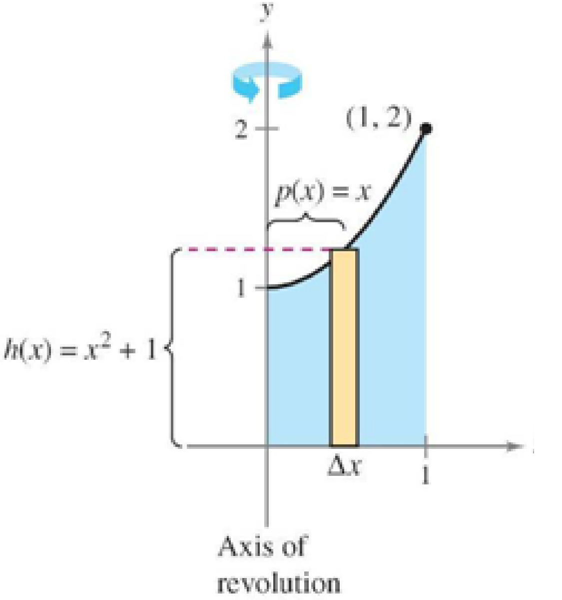
In Figure 6.33(b), you can see that the shell method requires only one integral to find the volume.

$$V = 2\pi \int_a^b p(x)h(x)\,dx$$ Apply shell method.

$$= 2\pi \int_0^1 x(x^2 + 1)\,dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1$$ Integrate.

$$= 2\pi\left(\frac{3}{4}\right) \quad = \frac{3\pi}{2}$$



Shell method

Figure 6.33(b)

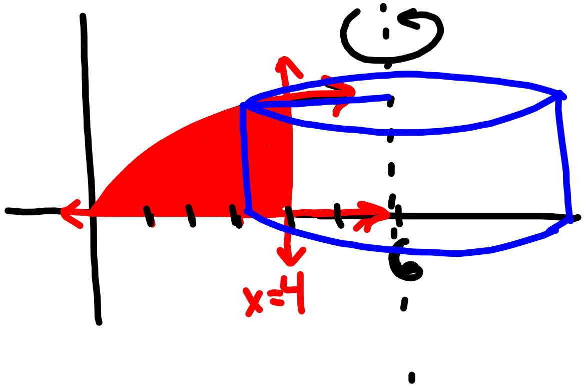
Rotating around a different line:

- When rotating around a different line (instead of an axis), the only thing that changes is the radius. The height and the limits stay the same.
- All you need to do then is change the radius to whatever the line is minus x (or y).

Rotating around a different line:

- Use the shell method to find the volume of the solid generated by revolving the plane region about the indicated line.

$y = \sqrt{x}, \quad y = 0, \quad x = 4, \quad \textit{about the line } x = 6$



radius is now $(6-x)$
and height is $(\sqrt{x} - 0)$ ← upper minus lower

$$2\pi\int_0^4 (6-x)(\sqrt{x})\,dx \approx 120.637$$