6 Applications of Integration

6.3

Volume: The Shell Method

Objectives

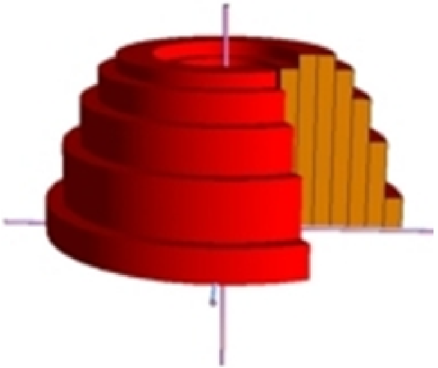
- Find the volume of a solid of revolution using the shell method.
- Compare the uses of the disk method and the shell method.

The Shell Method

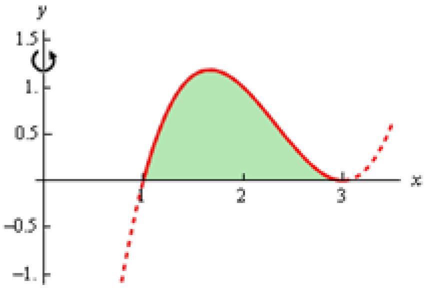
The Shell Method

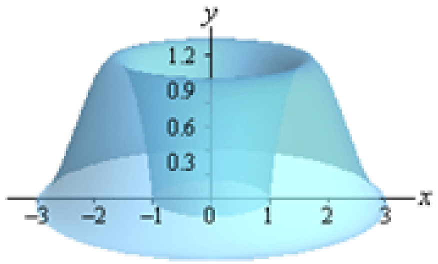
An alternative method for finding the volume of a solid of revolution is called the **shell method** because it uses cylindrical shells.

A comparison of the advantages of the disk and shell methods is given later in this section.

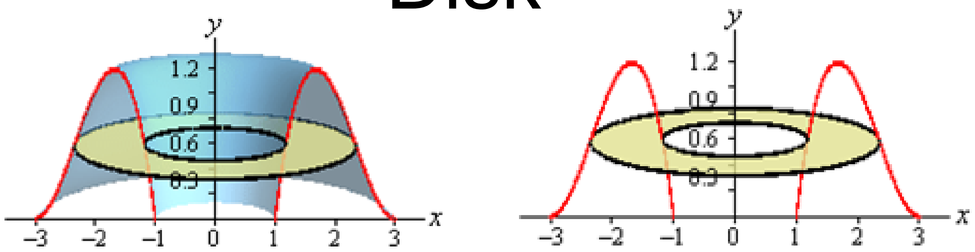


The Disk & Shell Methods

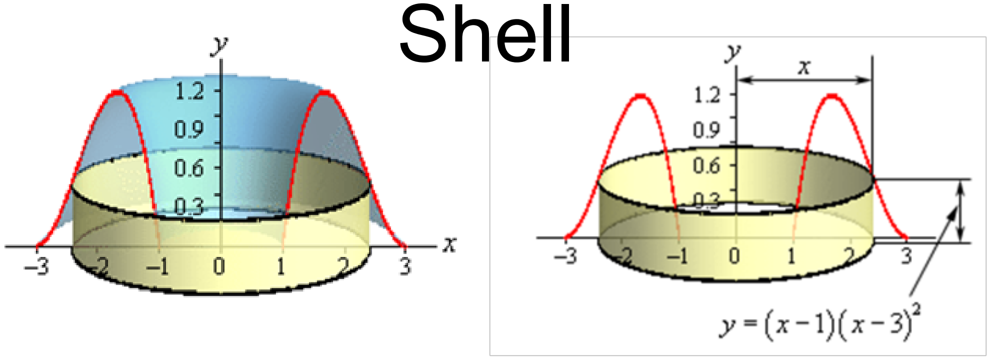




Disk



Shell



The Shell Method

Assume that the plane region in Figure 6.28 is revolved about a line to form the indicated solid.

If you consider a horizontal rectangle of width Δy, then, as the plane region is revolved about a line parallel to the *x*-axis, the rectangle generates a representative shell whose volume is

$$\Delta V = 2\pi[p(y)h(y)]\,\Delta y.$$

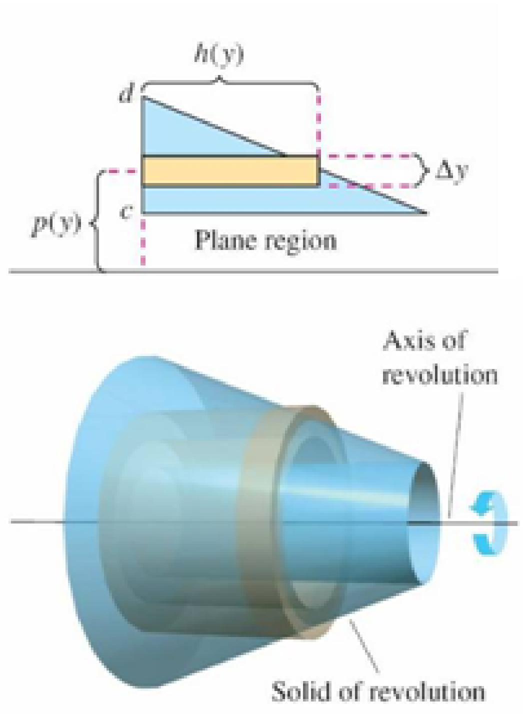


Figure 6.28

The Shell Method

You can approximate the volume of the solid by *n* such shells of thickness Δy, height *h*(*yi*), and average radius *p*(*yi*).

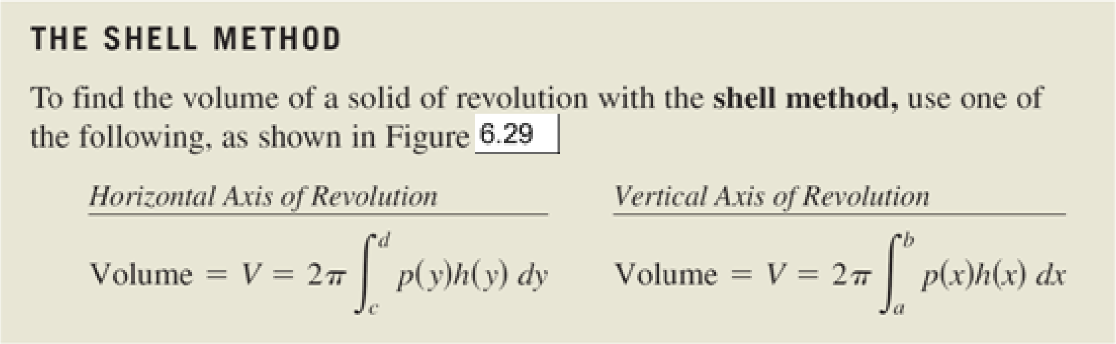
$$\text{Volume of solid} \approx \sum_{i=1}^{n} 2\pi[p(y_i)h(y_i)]\,\Delta y = 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

This approximation appears to become better and better as $\|\Delta\| \to 0 (n \to \infty)$. So, the volume of the solid is

$$\text{Volume of solid} = \lim_{\|\Delta\|\to 0} 2\pi\sum_{i=1}^{n}[p(y_i)h(y_i)]\,\Delta y$$

$$= 2\pi\int_{c}^{d}[p(y)h(y)]\,dy.$$

The Shell Method



THE SHELL METHOD

To find the volume of a solid of revolution with the **shell method,** use one of the following, as shown in Figure 6.29

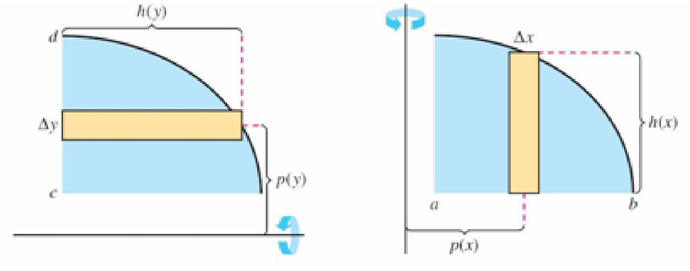
Horizontal Axis of Revolution

$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y)\, dy$$

Vertical Axis of Revolution

$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x)\, dx$$

radius: p(y) = y
height: h(y) = f(y)



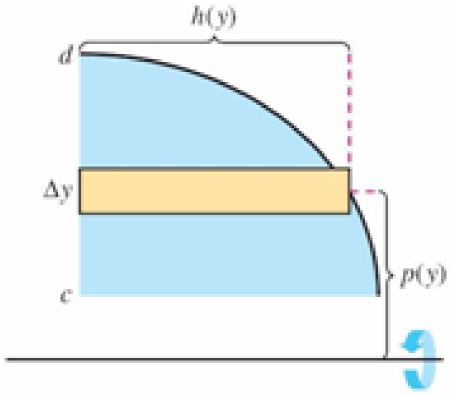
Horizontal axis of revolution

Vertical axis of revolution

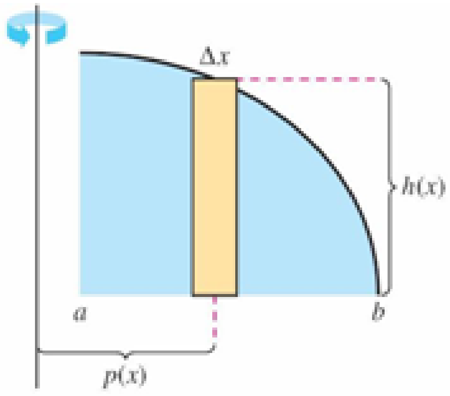
radius: p(x) = x
height: h(x) = f(x)

Figure 6.29

The Shell Method



Horizontal axis of revolution



Vertical axis of revolution

radius: p(y) = y

height: h(y) = f(y)

radius: p(x) = x

height: h(x) = f(x)

Example 1 – *Using the Shell Method to Find Volume*

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \leq x \leq 1$) about the y-axis.

Solution on next pages

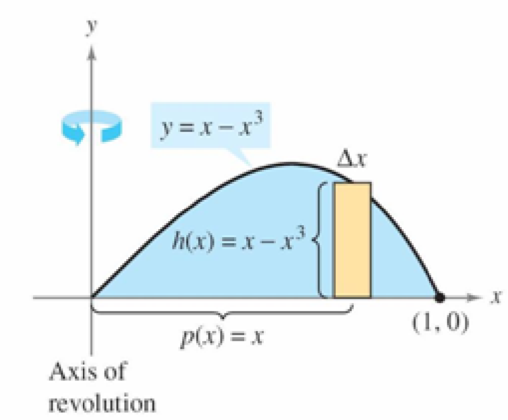


Figure 6.30

Example 1 – *Using the Shell Method to Find Volume*

Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis ($0 \leq x \leq 1$) about the y-axis.

Solution:

Because the axis of revolution is vertical, use a vertical representative rectangle, as shown in Figure 6.30. The width Δx indicates that x is the variable of integration.

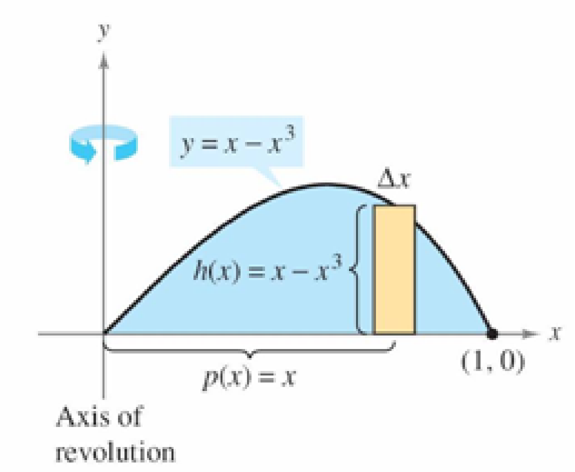


Figure 6.30

Example 1 – *Solution*

cont'd

The distance from the center of the rectangle to the axis of revolution is *p*(*x*) = *x*, and the height of the rectangle is *h*(*x*) = *x* – *x*3.

Because *x* ranges from 0 to 1, the volume of the solid is

$$\begin{aligned} V &= 2\pi \int_a^b \underset{\uparrow \text{ radius is } x}{\underline{p(x)}} h(x)\,dx = 2\pi \int_0^1 x(x - x^3)\,dx && \text{Apply shell method.} \\ &= 2\pi \int_0^1 (-x^4 + x^2)\,dx && \text{Simplify.} \\ &= 2\pi \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_0^1 && \text{Integrate.} \\ &= 2\pi \left(-\frac{1}{5} + \frac{1}{3} \right) \quad = \frac{4\pi}{15}. \end{aligned}$$

Comparison of Disk and Shell Methods

Shell Method -vs- Disk Method

Shell Method: Use the shell method if the equation is solved for the same variable as the axis that you are rotating around.
For example: volume of the area bounded by **y** = $-3x^2+8x$ and y=0 and rotated around the **y-axis** would be easiest using the shell method. (The upper and lower **limits** for the integrand are **on the axis you did not rotate around**.)

Disk Method: Use the disk method if the equation is solved for the variable that is not the axis you are rotating around.
For example: volume of the area bounded by **y** = $-3x^2+8x$ and y=0 and rotated around the **x-axis** would be easiest using the disk method. (The upper and lower **limits** for the integrand are **on the axis you rotated around**.)

	Disk/Washer Method	Shell Method
Rotate about x-axis	$\int \underline{\underline{x\text{'s}}}$ limits from x-axis	$\int y\text{'s}$ limits from y-axis
Rotate about y-axis	$\int y\text{'s}$ limits from y-axis	$\int \underline{\underline{x\text{'s}}}$ limit's from x-axis

Example 2: Find the volume of the area bounded by **y** = -3x²+12, y=0, & x=0 rotated about the **y-axis.** ↓

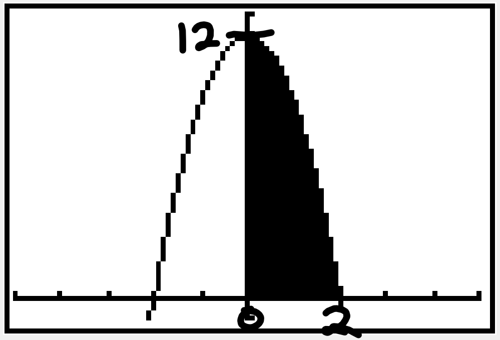
solved for x: $x=\sqrt{4-\frac{1}{3}y}$

Disk Method:

$$\pi\int_0^{12}\sqrt{4-\frac{1}{3}y}^{\,2}\,dy=\underline{\underline{24\pi}}$$

Shell Method:

$$2\pi\int_0^{2}x(-3x^2+12)\,dx=\underline{\underline{24\pi}}$$



Find the volume of the area bounded by **y** = -3x²+12, y=0 & x=0 rotated about the **x-axis.**

Disk Method:

$$\pi\int_0^{2}(-3x^2+12)^2\,dx=\underline{\underline{153.6\pi}}$$

Shell Method:

$$2\pi\int_0^{12}y\sqrt{4-\frac{1}{3}y}\,dy=\underline{\underline{153.6\pi}}$$

Example 3:

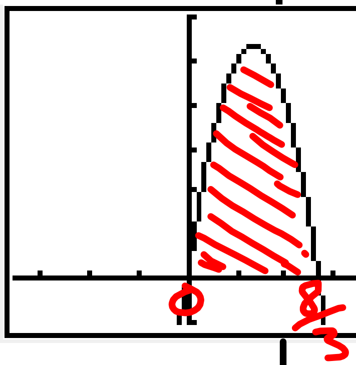
Find the volume of the area bounded by
y = $-3x^2+8x$ and y=0 rotated about the **y-axis.**

~~Disk Method:~~

can't solve for x, so we can't use this method.

Shell Method:

$$2\pi\int_0^{8/3} x(-3x^2+8x)\,dx \approx \underline{\underline{79.432}}$$



Find the volume of the area bounded by
y = $-3x^2+8x$ and y=0 rotated about the **x-axis**.

Disk Method:

$$\pi\int_0^{8/3}(-3x^2+8x)^2\,dx \approx \underline{\underline{127.091}}$$

~~Shell Method:~~

can't solve for x, so we can't use this method.

Comparison of Disk and Shell Methods

For the disk method, the representative rectangle is always *perpendicular* to the axis of revolution, whereas for the shell method, the representative rectangle is always *parallel* to the axis of revolution.

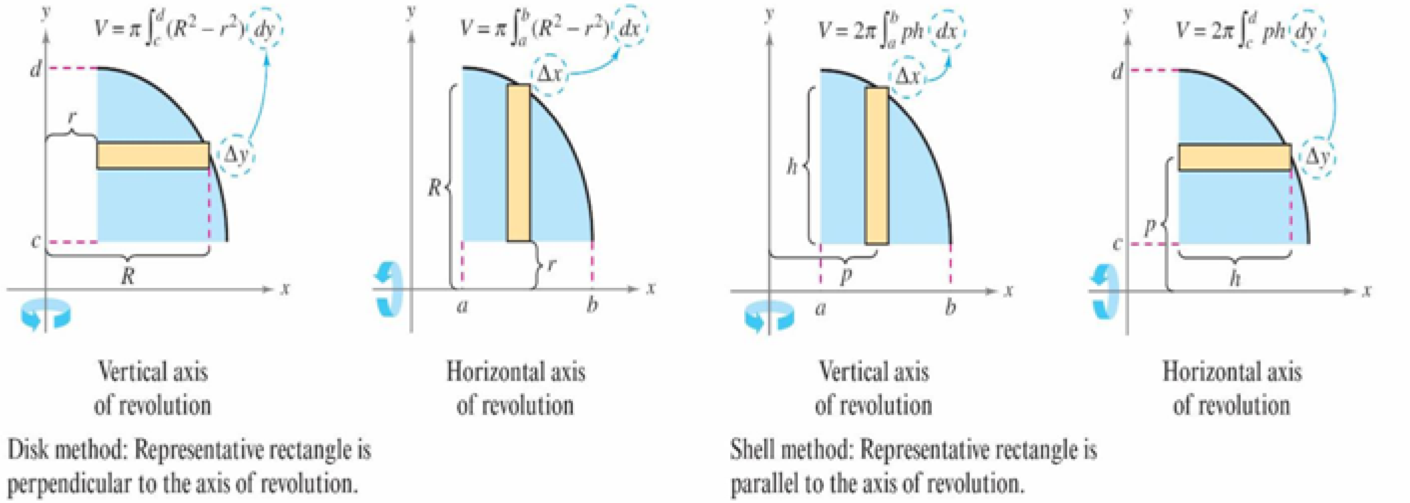
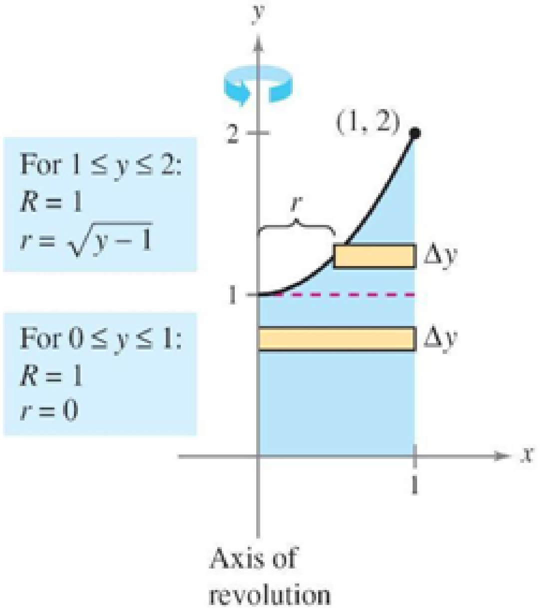


Figure 6.32

Example 4 – Which method is best?

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

The washer method requires two integrals to determine the volume of this solid. See Figure 6.33(a).



Disk method

Figure 6.33(a)

Example 4 Solution: Disk/Washer Method

$$V = \pi \int_0^1 (1^2 - 0^2)\,dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy$$

Apply washer method.

$$= \pi \int_0^1 1\,dy + \pi \int_1^2 (2 - y)\,dy$$

Simplify.

$$= \pi \Big[y\Big]_0^1 + \pi \left[2y - \frac{y^2}{2}\right]_1^2$$

Integrate.

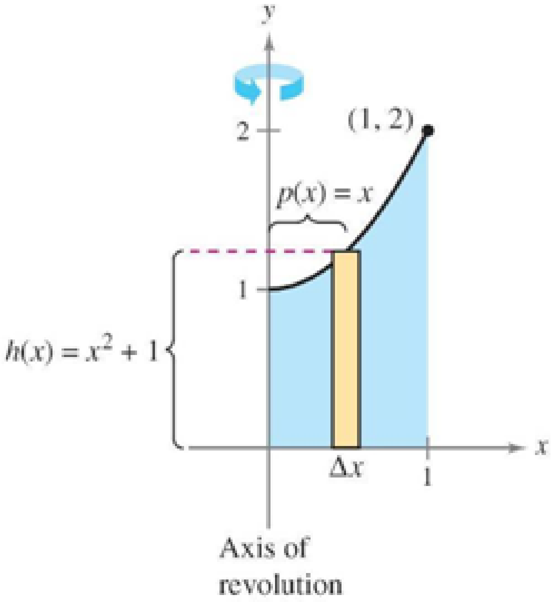
$$= \pi + \pi\left(4 - 2 - 2 + \frac{1}{2}\right)$$

$$= \frac{3\pi}{2}$$

Example 4: Use the Shell Method:

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

Solution on next page



Example 4 *Shell Method Solution*

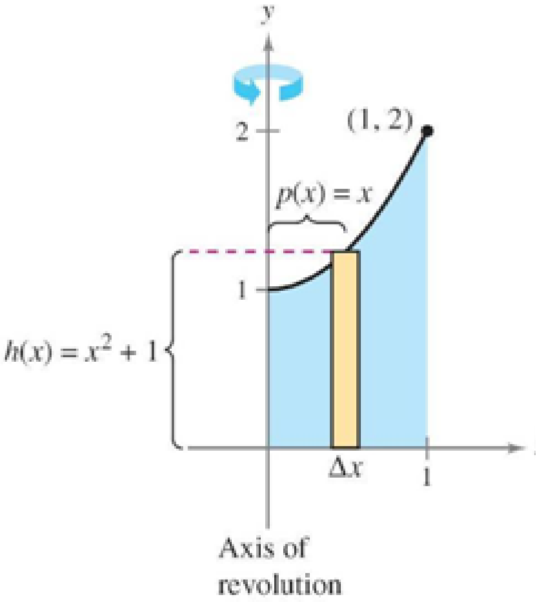
In Figure 6.33(b), you can see that the shell method requires only one integral to find the volume.

$$V = 2\pi \int_a^b p(x)h(x)\,dx$$ Apply shell method.

$$= 2\pi \int_0^1 x(x^2 + 1)\,dx$$

$$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1$$ Integrate.

$$= 2\pi\left(\frac{3}{4}\right) \quad = \frac{3\pi}{2}$$



Shell method

Figure 6.33(b)

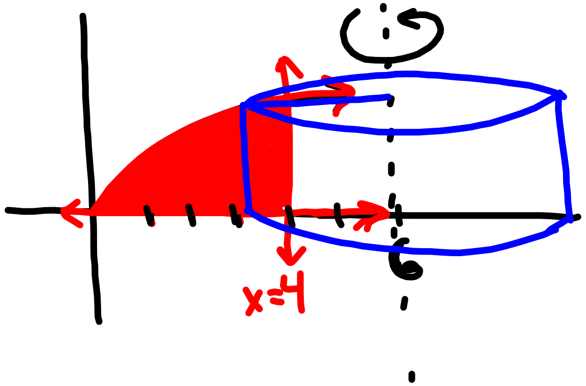
Rotating around a different line:

- When rotating around a different line (instead of an axis), the only thing that changes is the radius. The height and the limits stay the same.
- All you need to do then is change the radius to whatever the line is minus x (or y).

Rotating around a different line:

- Use the shell method to find the volume of the solid generated by revolving the plane region about the indicated line.

$y = \sqrt{x}$, $y = 0$, $x = 4$, *about the line* $x = 6$



radius is now $(6-x)$
and height is $(\sqrt{x} - 0)$ ← upper minus lower

$$2\pi \int_0^4 (6-x)(\sqrt{x})\, dx \approx \underline{\underline{120.637}}$$