Slope Fields and
Differential Equations
Objectives

- Review verifying solutions to differential equations.
- Review solving differential equations.
- Review using initial conditions to find particular solutions of differential equations.
- Use slope fields to approximate solutions of differential equations.
Verifying Solutions of Differential Equations
A **differential equation** in \(x\) and \(y\) is an equation that involves \(x\), \(y\), and derivatives of \(y\).

A function \(y = f(x)\) is called a **solution** of a differential equation if the equation is satisfied when \(y\) and its derivatives are replaced by \(f(x)\) and its derivatives.
Example 1 – Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$  
b. $y = 4e^{-x}$  
c. $y = Ce^x$

Solution:

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2\sin x \neq 0.$$ 

So, $y = \sin x$ is not a solution.
Example 1 – Solution

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$ 

So, $y = 4e^{-x}$ is a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$ 

So, $y = Ce^x$ is a solution for any value of $C$. 
General and Particular Solutions
Differentiation and substitution would show that \( y = e^{-2x} \) is a solution of the differential equation
\[
y' + 2y = 0.
\]

It can be shown that every solution of this differential equation is of the form
\[
y = Ce^{-2x}
\]
where \( C \) is any real number.

This solution is called the \textbf{general solution}. 
Show how to find the general solution of the differential equation $y' + 2y = 0$. 
General and Particular Solutions

*The order of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation.

*It can be shown that a differential equation of order $n$ has a general solution with $n$ arbitrary constants.

*Example: Find the general solution of the second-order differential equation $s''(t) = -32$. 
Particular solutions of a differential equation are obtained from initial conditions that give the values of the dependent variable or one of its derivatives for particular values of the independent variable.
The term “initial condition” stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$.

Example: Find the particular solution for the differential equation $s''(t) = -32$ having initial conditions $s(0) = 80$ and $s'(0) = 64$. 

General and Particular Solutions
General and Particular Solutions

Geometrically, the general solution of a first-order differential equation represents a family of curves known as solution curves, one for each value assigned to the arbitrary constant.

For instance, you can verify that every function of the form

\[ y = \frac{C}{x} \]

is a solution of the differential equation \( xy' + y = 0 \).
Figure 6.1 shows four of the solution curves corresponding to different values of $C$.

**Particular solutions** of a differential equation are obtained from initial conditions that give the values of the dependent variable or one of its derivatives for particular values of the independent variable.
Example 2 – Finding a Particular Solution

For the differential equation \( xy' - 3y = 0 \), verify that \( y = Cx^3 \) is a solution, and find the particular solution determined by the initial condition \( y = 2 \) when \( x = -3 \).

Solution:
You know that \( y = Cx^3 \) is a solution because \( y' = 3Cx^2 \) and

\[
xy' - 3y = x(3Cx^2) - 3(Cx^3) = 0.
\]
Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3$$

$$2 = C(-3)^3$$

$$-\frac{2}{27} = C$$

and you can conclude that the particular solution is

$$y = -\frac{2}{27} x^3.$$  

Try checking this solution by substituting for $y$ and $y'$ in the original differential equation.
Slope Fields
Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation.

Consider a differential equation of the form

\[ y' = F(x, y) \]

where \( F(x, y) \) is some expression in \( x \) and \( y \).

At each point \((x, y)\) in the \( xy\)-plane where \( F \) is defined, the differential equation determines the slope \( y' = F(x, y) \) of the solution at that point.
Slope Fields

If you draw short line segments with slope $F(x, y)$ at selected points $(x, y)$ in the domain of $F$, then these line segments form a slope field, or a direction field, for the differential equation $y' = F(x, y)$.

Each line segment has the same slope as the solution curve through that point.

A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.
Example 3 – Sketching a Slope Field

Sketch a slope field for the differential equation \( y' = x - y \)
for the points \((-1, 1), (0, 1), \) and \((1, 1)\).

Solution:
The slope of the solution curve at any point \((x, y)\) is
\[
F(x, y) = x - y.
\]

So, the slope at \((-1, 1)\) is \(y' = -1 - 1 = -2\), the slope at
\((0, 1)\) is \(y' = 0 - 1 = -1\), and the slope at \((1, 1)\) is
\(y' = 1 - 1 = 0\).
Example 3 – Solution

Draw short line segments at the three points with their respective slopes, as shown in Figure 6.2.

Figure 6.2
General and Particular Solutions

Make a slope field for the differential equation $xy’ + y = 0$.

Now find the general solution for the same equation.
See how the slope field represents all the general solutions of the differential equation.

Solution curves for \( xy' + y = 0 \)

**General solution:** \( y = \frac{C}{x} \)